Graph Theory

Lecture by Prof. Dr. Maria Axenovich

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Contents

Introduction

These notes include major definitions, theorems, and proofs for the graph theory course given by Prof. Maria Axenovich at KIT during the winter term 2019/20. Most of the content is based on the book "Graph Theory" by Reinhard Diestel [4]. A free version of the book is available at `http://diestel-graph-theory.com`.

Conventions:

- $G = (V, E)$ is an arbitrary (undirected, simple) graph
- $n := |V|$ is its number of vertices
- $m := |E|$ is its number of edges

Notation

notation	definition	meaning
$\binom{V}{k}$, V finite set, k integer	$\{S \subseteq V : \|S\| = k\}$	the set of all k-element subsets of V
V^2, V finite set	$\{(u, v) : u, v \in V,\ u \neq v\}$	the set of all ordered pairs of elements in V
$[n]$, n integer	$\{1, \ldots, n\}$	the set of the first n positive integers
$\mathbb{N}$	$1, 2, \ldots$	the natural numbers, not including 0
2^S, S finite set	$\{T : T \subseteq S\}$	the power set of S, i.e., the set of all subsets of S
$S \triangle T$, S, T finite sets	$(S \cup T) \setminus (S \cap T)$	the symmetric difference of sets S and T, i.e., the set of elements that appear in exactly one of S or T
$A \dot\cup B$, A, B disjoint sets	$A \cup B$	the disjoint union of A and B

1 Preliminaries

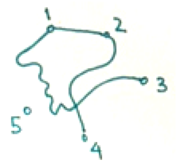
Definition 1.1. A *graph* G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$ is a set of pairs of elements in V. graph, G

- The set V is called the set of *vertices* and E is called the set of *edges* of G. vertex, edge
- The edge $e = \{u, v\} \in \binom{V}{2}$ is also denoted by $e = uv$.
- If $e = uv \in E$ is an edge of G, then u is called *adjacent* to v and u is called *incident* to e. adjacent, incident
- If e_1 and e_2 are two edges of G, then e_1 and e_2 are called *adjacent* if $e_1 \cap e_2 \neq \emptyset$, i.e., the two edges are incident to the same vertex in G.

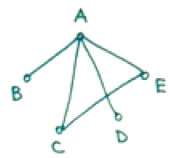
We can visualize graphs $G = (V, E)$ using pictures. For each vertex $v \in V$ we draw a point (or small disc) in the plane. And for each edge $uv \in E$ we draw a continuous curve starting and ending in the point/disc for u and v, respectively.

Several examples of graphs and their corresponding pictures follow:

$V = [5]$, $E = \{12, 13, 24\}$



$V = \{A, B, C, D, E\}$,
$E = \{AB, AC, AD, AE, CE\}$



Definition 1.2 (Graph variants)**.**

- A *directed graph* is a pair $G = (V, A)$ where V is a finite set and $A \subseteq V^2$. The edges of a directed graph are also called *arcs*. directed graph arc
- A *multigraph* is a pair $G = (V, E)$ where V is a finite set and E is a multiset of elements from $\binom{V}{1} \cup \binom{V}{2}$, i.e., we also allow loops and multiedges. multigraph
- A *hypergraph* is a pair $H = (X, E)$ where X is a finite set and $E \subseteq 2^X \setminus \{\emptyset\}$. hypergraph

Definition. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we say that G_1 and G_2 are *isomorphic*, denoted by $G_1 \simeq G_2$, if there exists a bijection $\phi : V_1 \to V_2$ with $xy \in E_1$ if and only if $\phi(x)\phi(y) \in E_2$. Loosely speaking, G_1 and G_2 are isomorphic if they are the same up to renaming of vertices. isomorphic, $\simeq$

When making structural comments, we do not normally distinguish between isomorphic graphs. Hence, we usually write $G_1 = G_2$ instead of $G_1 \simeq G_2$ whenever vertices $=$

are indistinguishable. Then we use the informal expression *unlabeled graph* (or just *graph* when it is clear from the context) to mean an isomorphism class of graphs. unlabeled graph

Important graphs and graph classes

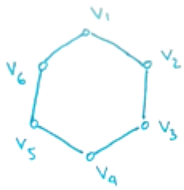
Definition. For all natural numbers n we define:

- the *complete graph* K_n on n vertices as the (unlabeled) graph isomorphic to $\left([n], \binom{[n]}{2}\right)$. We also call complete graphs *cliques*. complete graph, K_n

K_5

K_3

- for $n \geq 3$, the *cycle* C_n on n vertices as the (unlabeled) graph isomorphic to $\left([n], \{\{i, i+1\} : i = 1, \dots, n-1\} \cup \{n, 1\}\right)$. The *length of a cycle* is its number of edges. We write $C_n = 12\dots n1$. The cycle of length 3 is also called a *triangle*. cycle, C_n triangle



- the *path* P_n on n vertices as the (unlabeled) graph isomorphic to $\left([n], \{\{i, i+1\} : i = 1, \dots, n-1\}\right)$. The vertices 1 and n are called the *endpoints* or *ends* of the path. The *length of a path* is its number of edges. We write $P_n = 12\dots n$. path, P_n

- the *empty graph* E_n on n vertices as the (unlabeled) graph isomorphic to $([n], \emptyset)$. Empty graphs correspond to independent sets. empty graph, E_n

E_{10}

- for $m \geq 1$, the *complete bipartite graph* $K_{m,n}$ on $n+m$ vertices as the (unlabeled) graph isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$, where $|A| = m$ and $|B| = n$, $A \cap B = \emptyset$.

  complete bipartite graph, $K_{m,n}$

$K_{4,2}$

- for $r \geq 2$, a *complete r-partite* graph as an (unlabeled) graph isomorphic to

  complete r-partite

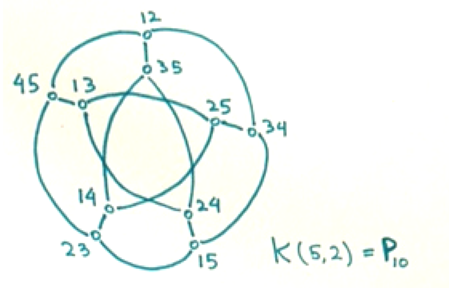
$$\big(A_1 \dot\cup \cdots \dot\cup A_r, \{xy : x \in A_i, y \in A_j, i \neq j\}\big),$$

where $A_1, \ldots, A_r$ are non-empty finite sets. In particular, the complete bipartite graph $K_{m,n}$ is a complete 2-partite graph.

- the *Petersen graph* as the (unlabeled) graph isomorphic to

  Petersen graph

$$\left(\binom{[5]}{2}, \{\{S,T\} : S,T \in \binom{[5]}{2}, S \cap T = \emptyset\}\right).$$



- for a natural number k, $k \leq n$, the *Kneser graph* $K(n,k)$ as the (unlabeled) graph isomorphic to

  Kneser graph, $K(n,k)$

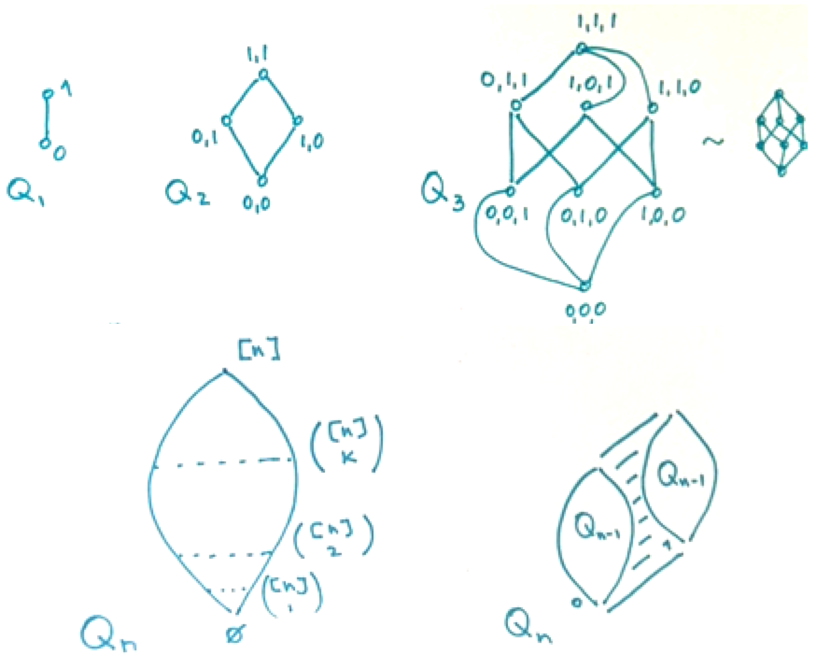
$$\left(\binom{[n]}{k}, \left\{\{S,T\} : S,T \in \binom{[n]}{k}, S \cap T = \emptyset\right\}\right).$$

Note that $K(5,2)$ is the Petersen graph.

- the *n-dimensional hypercube* Q_n as the (unlabeled) graph isomorphic to hypercube, Q_n

$$\left(2^{[n]}, \{\{S,T\} : S,T \in 2^{[n]}, |S \triangle T| = 1\}\right).$$

  Vertices are labeled either by corresponding sets or binary indicators vectors. For example the vertex $\{1,3,4\}$ in Q_6 is coded by $(1,0,1,1,0,0,0)$.



Basic graph parameters and degrees

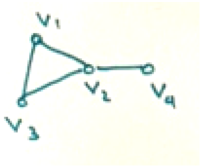
Definition 1.3. Let $G = (V,E)$ be a graph. We define the following parameters of G.

- The graph G is *non-trivial* if it contains at least one edge, i.e., $E \neq \emptyset$. Equivalently, G is non-trivial if G is not an empty graph. non-trivial
- The *order of* G, denoted by $|G|$, is the number of vertices of G, i.e., $|G| = |V|$. order, $|G|$
- The *size of* G, denoted by $\|G\|$, is the number of edges of G, i.e., $\|G\| = |E|$. Note that if the order of G is n, then the size of G is between 0 and $\binom{n}{2}$. size, $\|G\|$
- Let $S \subseteq V$ be a set of vertices. The *neighbourhood of* S, denoted by $N(S)$, is the set of vertices in V that have an adjacent vertex in S. The elements of $N(S)$ are called *neighbours* of S. Instead of $N(\{v\})$ for $v \in V$ we usually write $N(v)$. neighbourhood, $N(v)$ neighbour

- If the vertices of G are labeled $v_1, \dots, v_n$, then there is an $n \times n$ matrix A with entries in $\{0, 1\}$, which is called the *adjacency matrix* and is defined as follows:

adjacency matrix

$$v_i v_j \in E \qquad \Leftrightarrow \qquad A[i, j] = 1$$

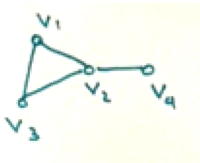


$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A graph and its adjacency matrix.

- The *degree* of a vertex v of G, denoted by $d(v)$ or $\deg(v)$, is the number of edges incident to v.

degree, $d(v)$



$\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 2$, $\deg(v_4) = 1$

- A vertex of degree 1 in G is called a *leaf*, and a vertex of degree 0 in G is called an *isolated vertex*.

leaf
isolated vertex

- The *degree sequence* of G is the multiset of degrees of vertices of G, e.g. in the example above the degree sequence is $\{1, 2, 2, 3\}$.

degree sequence

- The *minimum degree of* G, denoted by $\delta(G)$, is the smallest vertex degree in G (it is 1 in the example).

minimum degree, $\delta(G)$

- The *maximum degree of* G, denoted by $\Delta(G)$, is the highest vertex degree in G (it is 3 in the example).

maximum degree, $\Delta(G)$

- The graph G is called k-*regular* for a natural number k if all vertices have degree k. Graphs that are 3-regular are also called *cubic*.

regular
cubic

- The *average degree of* G is defined as $d(G) = \left(\sum_{v \in V} \deg(v)\right)/|V|$. Clearly, we have $\delta(G) \le d(G) \le \Delta(G)$ with equality if and only if G is k-regular for some k.

average degree, $d(G)$

Lemma 1 (Handshake Lemma, 1.2.1)**.** For every graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v).$$

Proof. Let $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$. Then

$$|X| = \sum_{v \in V(G)} d(x)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|.$$

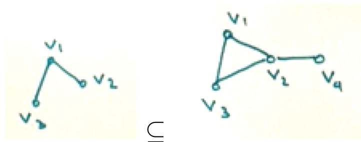
The result follows. □

Corollary 2. The sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Subgraphs

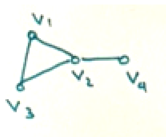
Definition 1.4.

- A graph $H = (V', E')$ is a *subgraph of* G, denoted by $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If H is a subgraph of G, then G is called a *supergraph of* H, denoted by $G \supseteq H$. In particular, $G_1 = G_2$ if and only if $G_1 \subseteq G_2$ and $G_1 \supseteq G_2$. subgraph, $\subseteq$ supergraph, $\supseteq$



- A subgraph H of G is called an *induced subgraph* of G if for every two vertices $u, v \in V(H)$ we have $uv \in E(H) \Leftrightarrow uv \in E(G)$. In the example above H is not an induced subgraph of G. Every induced subgraph of G can be obtained by deleting vertices (and all incident edges) from G. induced subgraph

  Examples:





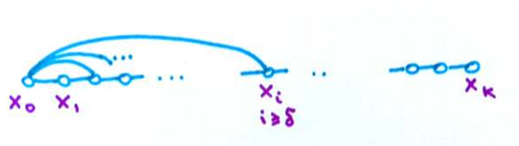
- Every induced subgraph of G is uniquely defined by its vertex set. We write $G[X]$ for the induced subgraph of G on vertex set X, i.e., $G[X] = \big(X, \{xy : x, y \in X,\ xy \in E(G)\}\big)$. Then $G[X]$ is called the *subgraph of G induced by the vertex set $X \subseteq V(G)$*. $G[X]$

  Example: G and $G[\{1, 2, 3, 4\}]$: 

- If H and G are two graphs, then an *(induced) copy* of H in G is an (induced) subgraph of G that is isomorphic to H. copy
- A subgraph $H = (V', E')$ of $G = (V, E)$ is called a *spanning subgraph* of G if $V' = V$. spanning subgraph
- A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. In this case, the vertex set can be written as $V = A \dot\cup B$ such that $E \subseteq \{ab \mid a \in A, b \in B\}$. The sets A and B are called *partite sets of G*. bipartite, partite sets
- A *cycle (path, clique)* in G is a subgraph H of G that is a cycle (path, complete graph). clique
- An *independent set* in G is an induced subgraph H of G that is an empty graph. independent set
- A *walk* (of length k) is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. walk, closed walk
- Let $A, B \subseteq V$. A path P in G is called an *A-B-path* if $P = v_1 \ldots v_k$, $V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_k\}$. When $A = \{a\}$ and $B = \{b\}$, we simply call P an a-b-path. If G contains an a-b-path we say that the vertices a and b are *linked by a path*. A-B-path
- Two paths P, P' in G are called *independent* if every vertex contained in both P and P' (if any) is an endpoint of P and P'. I.e., P and P' can share only endpoints. independent paths
- A graph G is called *connected* if any two vertices are linked by a path. connected
- A subgraph H of G is *maximal*, respectively *minimal*, with respect to some property if there is no supergraph, respectively subgraph, of H with that property. maximal, minimal
- A maximal connected subgraph of G is called a *connected component* of G. component
- A graph G is called *acyclic* if G does not have any cycle. Acyclic graphs are also called *forests*. acyclic, forest
- A graph G is called a *tree* if G is connected and acyclic. tree

Proposition 3. If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G)+1$ vertices.

Proof. Let $P = (x_0, \ldots, x_k)$ be a longest path in G. Then $N(x_0) \subseteq V(P)$, otherwise $(x, x_0, x_1, \ldots, x_k)$ is a longer path, for $x \in N(x_0) \setminus V(P)$. Let i be the largest index such that $x_i \in N(x_0)$, then $i \geq |N(x_0)| \geq \delta$. So, $(x_0, x_1, \ldots, x_i, x_0)$ is a cycle of length at least $\delta(G)+1$.



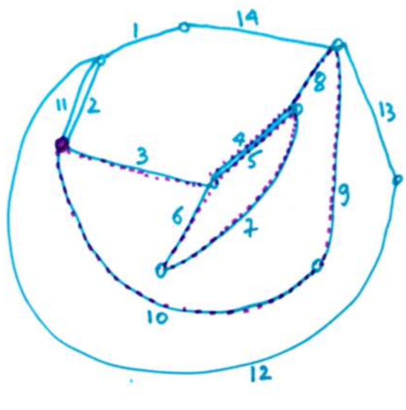
□

Proposition 4. If for distinct vertices u and v a graph has a u-v-walk, then it has a u-v-path.

Proof. Consider a u-v-walk W with the smallest number of edges. Assume that W does not form a path, then there is a repeated vertex, w, i.e.,

$$W = u, e, v_1, e_1, \ldots, e_k, w, e_{k+1}, \ldots, e_\ell, w, e_{\ell+1}, \ldots, v.$$

Then $W_1 = u, e, v_1, \ldots, e_k, w, e_{\ell+1}, \ldots, v$ is a shorter u-v-walk, a contradiction. □



Proposition 5. If a graph has a closed walk of odd length, then it contains an odd cycle.

Proof. Let W be a closed odd walk of the smallest length. If it is a cycle, we are done. Otherwise there is a repeated vertex, so W is an edge-disjoint union of two closed walks. Since the sum of the lengths of these walks is odd, one of them is an odd closed walk with length strictly less that the length of W. A contradiction to the minimality of W. $\square$

Proposition 6. If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proof. Let W be a shortest closed walk with a non-repeated edge e. If W is a cycle, we are done. Otherwise, there is a repeated vertex and W is a union of two closed walks W_1 and W_2 that are shorter than W. One of them, say W_1, contains e, a non-repeated edge. This contradicts the minimality of W. $\square$

Proposition 1.5. A graph is bipartite if and only if it has no cycles of odd length.

Proof. Assume that G is a bipartite graph with parts A and B. Then any cycle has a form $a_1, b_1, a_2, b_2, \dots, a_k, b_k, a_1$, where $a_i \in A$, $b_i \in B$, $i \in [k]$. Thus any cycle has even length.

Now assume that G does not have cycles of odd length. We shall prove that G is bipartite. We can assume that G is connected, because otherwise we can treat the connected components separately. Let $v \in V(G)$. Let $A = \{u \in V(G) : dist(u, v) \equiv 0 \pmod 2\}$. Let $B = \{u \in V(G) : dist(u, v) \equiv 1 \pmod 2\}$. We claim that G is bipartite with parts A and B. To verify that it is sufficient to prove that A and B are independent sets. Let $u_1u_2 \in E(G)$. Let P_1 be a shortest u_1-v-path and P_2 be a shortest u_2-v-path. Then the union of P_1, P_2 and u_1u_2 forms a closed walk W. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$, then W is a closed odd walk. Thus G contains an odd cycle, a contradiction. Thus for any edge u_1u_2, u_1 and u_2 are in different parts A or B. Thus A and B are independent sets. $\square$

Theorem 1.6 (Eulerian Tour Condition, 1.8.1)**.** A connected graph has an Eulerian tour if and only if every vertex has even degree.

Proof. Assume that G is connected and has an Eulerian tour. Then by the definition of the tour, there is an even number of edges incident to each vertex.

On the other hand, assume that G is a connected graph with all vertices of even degree. Consider a walk $W = v_0, e_0, \dots, v_k$ with non-repeated edges and having largest possible number of edges.

First we show that W has to be a closed walk. Otherwise, if $v_0 \neq v_k$, we see that v_0 is incident to an odd number of edges in W. Since degree of v_0 is even, there is a vertex y such that the edge $e = v_0y$ is not in W, thus a walk $y, e, v_0, e_0, \dots, v_k$ obtained by extending W with an edge e is longer than W and also has no repeated edges, a contradiction. Thus $v_0 = v_k$.

Now we show that W contains all the edges of G. Otherwise, using connectivity of G, we see that there is an edge $e = x_iz$ of G that is incident to a vertex v_i of W and is not contained in W. Then the walk $x, e, v_i, e_i, v_{i+1}, \ldots, v_k, e_0, v_1, e_1, \ldots, v_i$ is a longer than W, a contradiction.

Therefore W is a closed walk that contains all the edges of the graph, i.e. W is an Eulerian tour. □

Lemma 7. Every tree on at least two vertices has a leaf.

Proof. If a tree T on at least two vertices does not have leaves then all vertices have degree at least 2, so there is a cycle in T, a contradiction. □

Lemma 8. A tree of order $n \geq 1$ has exactly $n-1$ edges.

Proof. We prove the statement by induction on n. When $n = 1$, there are no edges. Assume that each tree on $n = k$ vertices has $k-1$ edges, $k \geq 1$. Let's prove that each tree on $k+1$ vertices has k edges. Consider a tree T on $k+1$ vertices. Since $k+1 \geq 2$, T has a leaf, v. Let $T' = T - \{v\}$. We see that T' is connected because any u-w-path in T, for $u \neq v$ and $w \neq v$, does not contain v. We see also that T' is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus T' is a tree on k vertices. By induction $|E(T')| = k-1$. Thus $|E(T)| = |E(T')|+1 = (k-1)+1 = k$. □

Lemma 9. Every connected graph contains a spanning tree.

Proof. Let G be a connected graph. Consider T, an acyclic spanning subgraph of G with largest number of edges. If it is a tree, we are done. Otherwise, T has more than one component. Consider vertices u and v from different components of G. Consider a shortest u-v-path, P, in G. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then $T \cup \{e\}$ is acyclic. Indeed, if there were to be a cycle, it would contain e, however there is no y-x-path in $T \cup \{e\}$ except for xy. Thus $T \cup \{e\}$ is a spanning acyclic subgraph of G with more edges than T, a contradiction. □

Lemma 10. A connected graph on $n \geq 1$ vertices and $n-1$ edges is a tree.

Proof. HW □

Lemma 11. The vertices of every connected graph on $n \geq 2$ vertices can be ordered $(v_1, \ldots, v_n)$ so that for every $i \in \{1, \ldots, n\}$ the graph $G\big[\{v_1, \ldots, v_i\}\big]$ is connected.

Proof. Let G be a connected graph on n vertices. It contains a spanning tree T. Let v_n be a leaf of T, let v_{n-1} be a leaf of $T - \{v_n\}$, v_{n-2} be a leaf of $T - \{v_n, v_{n-1}\}$, and so on, v_k be a leaf in $T - \{v_n, v_{n-1}, \ldots, v_{k+1}\}$, $k = 2, \ldots, n$. Since deleting a leaf does not disconnect a tree, all the resulting graphs form spanning trees of $G\big[\{v_1, \ldots, v_i\}\big]$, $i = 1, \ldots, n$. A graph H having a spanning tree or any connected spanning subgraph H' is connected because a u-v-path in H' is a u-v-path in H. This observation completes the proof. □

Proposition 1.7. For any graph $G = (V, E)$ the following are equivalent:

(i) G is a tree, that is, G is connected and acyclic.

(ii) G is connected, but for any $e \in E$ in G the graph $G - e$ is not connected.

(iii) G is acyclic, but for any $x, y \in V(G)$, $xy \notin E(G)$ the graph $G + xy$ has a cycle.

(iv) G is connected and 1-degenerate.

(v) G is connected and $|E| = |V| - 1$.

(vi) G is acyclic and $|E| = |V| - 1$.

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

(viii) Any two vertices are joined by a unique path in G.

Proof. We give the proof of two implications. The rest is HW.

(i) $\Rightarrow$ (iii):
Let G be a tree, let's prove that G is acyclic, but for any $xy \notin E$ the graph $G + xy$ has a cycle. By the definition G is acyclic. Consider $x, y \in V(G)$ such that $xy \notin E(G)$. Since G is connected, there is an x-y-path P in G. Then $P \cup \{e\}$ is a cycle for $e = xy$.

(iii) $\Rightarrow$ (i):
Assume that G is acyclic, but for any $xy \notin E(G)$ the graph $G + xy$ has a cycle. Let's prove that G is a tree. It is given that G is acyclic, so we only need to prove that G is connected. Assume otherwise that there is no x-y-path in G for some two vertices x and y. Then in particular $xy \notin E(G)$. However, $G \cup \{xy\}$ has a cycle C and this cycle must contain the edge xy. Thus there are two edge-disjoint x-y-paths, one of which does not contain the edge xy and thus is a path in G. So, there is an x-y-path in G, a contradiction.

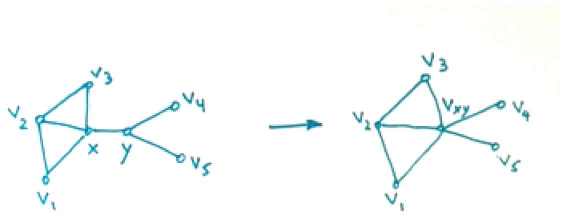
$\square$

Operations on graphs

Definition 1.8. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, $U \subseteq V$ be a subset of vertices of G and $F \subseteq \binom{V}{2}$ be a subset of pairs of vertices of G. Then we define

- $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. Note that $G, G' \subseteq G \cup G'$ and $G \cap G' \subseteq G, G'$. Sometimes, we also write $G + G'$ for $G \cup G'$. $G \cup G', G \cap G'$
- $G - U := G[V \setminus U]$, $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. If $U = \{u\}$ or $F = \{e\}$ then we simply write $G - u$, $G - e$ and $G + e$ for $G - U$, $G - F$ and $G + F$, respectively. $G - U$, $G - F$, $G + F$

- For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge* e. $G \circ e$ contract



- The *complement of* G, denoted by $\overline{G}$ or G^C, is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular, $G + \overline{G}$ is a complete graph, and $\overline{G} = (G + \overline{G}) - E$. complement, $\overline{G}$

More graph parameters

Definition 1.9. Let $G = (V, E)$ be any graph.

- The *girth of* G, denoted by $g(G)$, is the length of a shortest cycle in G. If G is acyclic, its girth is said to be ∞. girth, $g(G)$
- The *circumference* of G is the length of a longest cycle in G. If G is acyclic, its circumference is said to be 0. circumference
- The graph G is called *Hamiltonian* if G has a spanning cycle, i.e., there is a cycle in G that contains every vertex of G. In other words, G is Hamiltonian if and only if its circumference is $|V|$. Hamiltonian
- The graph G is called *traceable* if G has a spanning path, i.e., there is a path in G that contains every vertex of G. traceable
- For two vertices u and v in G, the *distance between* u *and* v, denoted by $d(u, v)$, is the length of a shortest u-v-path in G. If no such path exists, $d(u, v)$ is said to be ∞. distance, $d(u, v)$
- The *diameter of* G, denoted by $\operatorname{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e. diameter, $\operatorname{diam}(G)$

$$\operatorname{diam}(G) = \max_{u,v \in V} d(u, v).$$

- The *radius of* G, denoted by $\operatorname{rad}(G)$, is defined as radius, $\operatorname{rad}(G)$

$$\operatorname{rad}(G) = \min_{u \in V} \max_{v \in V} d(u, v).$$

- If there is a vertex ordering $v_1, \dots, v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leq d,$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is d-degenerate is called the *degeneracy* of G. d-degenerate degeneracy

We remark that the 1-degenerate graphs are precisely the forests.

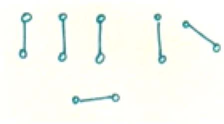
- A *proper k-edge colouring* is an assignment $c' \colon E \to [k]$ of colours in $[k]$ to edges such that no two adjacent edges receive the same colour. The *chromatic index of G*, or *edge-chromatic number*, is the minimal k such that G has a k-edge colouring. It is denoted by $\chi'(G)$. proper edge colouring chromatic index, $\chi'(G)$

- A *proper k-vertex colouring* is an assignment $c \colon V \to [k]$ of colours in $[k]$ to vertices such that no two adjacent vertices receive the same colour. The *chromatic number of G* is the minimal k such that G has a k-vertex colouring. It is denoted by $\chi(G)$. proper vertex colouring chromatic number, $\chi(G)$

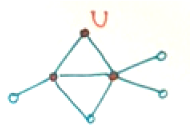
2 Matchings

Definition 2.1.

- A *matching* is a 1-regular graph, i.e., a matching is a graph M so that $E(M)$ is a union of pairwise non-adjacent edges and $2|E(M)| = |V(M)|$. matching



- A *matching in* G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\nu(G)$. $\nu(G)$
- A *vertex cover in* G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$. vertex cover $\tau(G)$



- A k*-factor of* G is a k-regular spanning subgraph of G. k-factor
- A *1-factor of* G is also called a *perfect matching* since it is a matching of largest possible size in a graph of order $|V|$. Clearly, G can only contain a perfect matching if $|V|$ is even. perfect matching

Theorem 2.2 (Hall's Marriage Theorem, 2.1.2)**.** Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

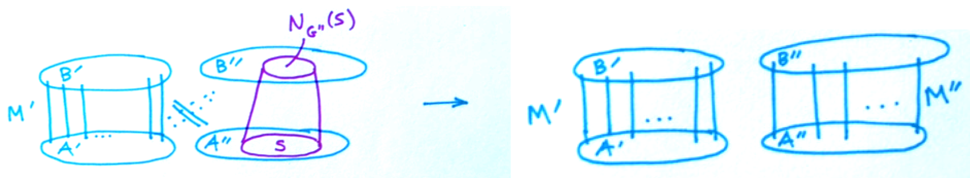
Proof. If G has a matching M containing all vertices of A, then for any $S \subseteq A$, $N(S)$ in G is at least as large as $N(S)$ in M, thus $|N(S)| \geq |S|$.

We say that the Hall's condition holds for a bipartite graph with parts A and B if $|N(S)| \geq |S|$ for all $S \subseteq A$. We shall prove by induction on $|A|$ that any bipartite graph with parts A and B satisfying Hall's condition has a matching containing all vertices of A, in other words, saturating A.

When $|A| = 1$, there is at least one edge in G and thus a matching saturating A.
Assume that the statement is true for all graphs G satisfying Hall's condition and with $|A| = k \geq 1$. Consider bipartite graph G with $|A| = k + 1$ and satisfying Hall's condition.

Case 1: $|N(S)| \geq |S| + 1$ for any $S \subseteq A$, $S \neq A$.
Let $G' = G - \{x, y\}$, for some edge xy, i.e., G' is obtained from G by deleting vertices $x \in A$ and $y \in B$, G' has parts $A' = A - \{x\}$ and $B' = B - \{y\}$. For any $S \subseteq A'$, $|N_{G'}(S')| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$. Thus G' satisfies Hall's condition and by induction has a matching M' saturating A'. Then $M = M' \cup \{xy\}$ is a matching in G saturating A.

Case 2: $|N(S_1)| = |S_1|$ for some $S_1 \subseteq A$, $S_1 \neq A$.
Let $A' = S_1$, $B' = N(A')$, $G' = G[A' \cup B']$. Since $|A'| < |A|$, and G' satisfies Hall's condition, G' has a matching M' saturating A' by induction. Now, consider $A'' = A - A'$, $B'' = B - B'$, $G'' = G[A'' \cup B'']$. We claim that G'' also satisfies Hall's condition. Assume not, and there is $S \subseteq A''$ such that $|N_{G''}(S)| < |S|$. Then $|N_G(S \cup A')| = |B' \cup N_{G''}(S)| = |B'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S|$, a contradiction to Hall's condition. Thus G'' does satisfy Hall's condition and there is a matching M'' saturating A'' in G''. Thus $M' \cup M''$ is a matching saturating A in G.



□

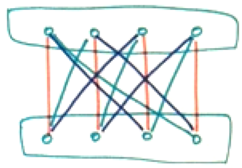
Corollary 12. Let G be a bipartite graph with partite sets A and B such that $|N(S)| \geq |S| - d$ holds for all $S \subseteq A$, and for a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.

Proof. Let $G = (A \cup B, E)$, let $G' = (A \cup B' \cup B, E \cup \{\{b', a\}, b' \in B', a \in A\})$, such that $B' \cap B = \emptyset$ and $|B'| = d$. Then for any $S \subseteq A$, $|N'_G(S)| \geq |N_G(S)| + d \geq |S| - d + d = |S|$. Thus G' satisfies Hall's condition and thus has a matching M saturating A, so $|E(M)| = |A|$. Consider $M' = M[A \cup B]$, then $|M'| \geq |M| - d = |A| - d$. □

Corollary 13. If G is a regular bipartite graph, it has a perfect matching.

Proof. Let $k \in \mathbb{N}$ and let G be a k-regular bipartite graph with parts A and B. Then $|E(G)| = k|A| = k|B|$, and thus $|A| = |B|$. Consider $S \subseteq A$, let e be the number of edges between S and $N(S)$. On one hand, $e = |S|k$, on the other hand $e \leq |N(S)|k$. Thus $|N(S)| \geq |S|$ and by Hall's theorem there is a matching saturating A. Since $|A| = |B|$, it is a perfect matching. □

Corollary 14. A k-regular bipartite graph has a proper k-edge-coloring.



Theorem 2.3 (Kőnig's Theorem)**.** Let G be bipartite. Then the size of a largest matching is the same as the size of a smallest vertex cover.

Proof. Let c be the vertex-cover number of G and m be the size of a largest matching of G. Since a vertex cover should contain at least one vertex from each matching edge, $c \geq m$.

Now, we shall prove that $c \leq m$. Let M be a largest matching in G, we need to show that $c \leq |M|$. Let A and B be the partite sets of G. An *alternating path* is a path that starts with a vertex in A not incident to an edge of M, and alternates between edges not in M and edges in M. Note that if an alternating path must end in a vertex saturated by M, otherwise one can find a larger matching.

Let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\},$$

$$U = U' \cup \{a : \ ab \in E(M), b \notin U'\}.$$

We see that $|U| = m$. We shall show that U is a vertex cover, i.e. that every edge of G contains a vertex from U. Indeed, if $ab \in E(M)$, then either a or b is in U. If $ab \notin E(M)$, we consider the following cases:

Case 0: $a \in U$. We are done.
Case 1: a is not incident to M. Then ab is an alternating path. If b is also not incident to M then $M \cup \{ab\}$ is a larger matching, a contradiction. Thus b is incident to M and then $b \in U$.
Case 2: a is incident to M. Then $ab' \in E(M)$ for some b'. Since $a \notin U$, we have that $b' \in U$, thus there is an alternating path P ending in b'. If P contains b, then $b \in U$, otherwise $Pb'ab$ is an alternating path ending in b, so $b \in U$. □

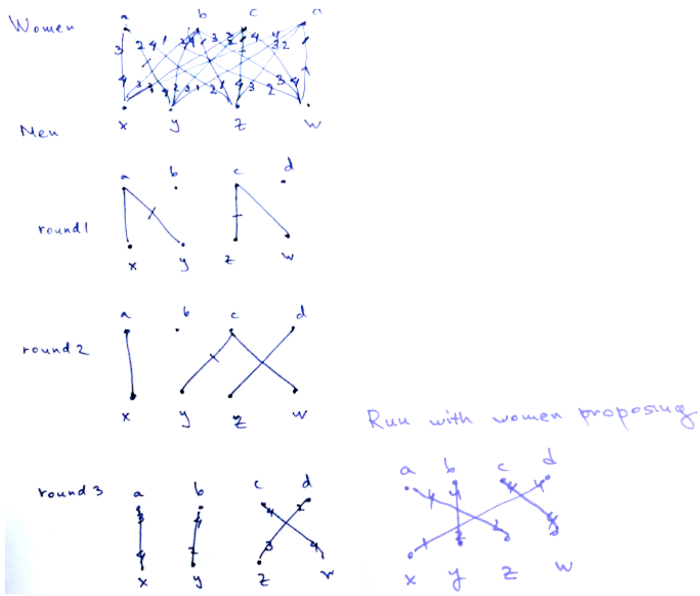
Assume that each vertex in a complete bipartite graph $G = (A \cup B, E)$ gives an ordering or a ranking to its neighbors, and write $y <_x y'$ if a vertex x "likes" y' more than y. A matching M in G is called *s*table matching if for any edge $e \notin M$ there is an edge $f \in M$ such that $f \cap e = x$, $f = xy$, $e = xy'$ and $y >_x y'$. If we assume for simplicity that A is a set of women and B is a set of men, then a stable matching is thought of as a set of “stable” marriages. I.e., for any ”marriage” from M, one of the spouses “has no reason to leave”.

Gale and Shapley proved in 1962 that there is always a stable matching in a bipartite graph equipped with a ranking of the neighbors for each vertex. They gave an algorithm to find one.

The algorithm is as follows: Initially, no one is engaged. During each round, each man who is not engaged proposes to highest on his list woman who did not reject him yet; for a woman receiving multiple proposals, she says ”maybe” to the highest ranked offer and rejects other proposals, the man to whom she said ”maybe” is now engaged to her, the rejected men are not engaged anymore. The rounds repeat until everybody is engaged.

“ Everyone gets married”:
At the end, there cannot be a man and a woman both unengaged, as he must have proposed to her at some point (since a man will eventually propose to everyone, if necessary) and, being proposed to, she would necessarily be engaged (to someone) thereafter.

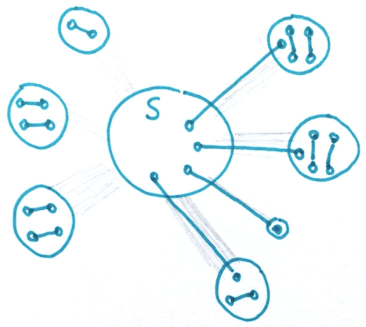
“The marriages are stable”:
Let Alice and Bob both be engaged, but not to each other. Upon completion of the algorithm, it is not possible for both Alice and Bob to prefer each other over their current partners. If Bob prefers Alice to his current partner, he must have proposed to Alice before he proposed to his current partner. If Alice accepted his proposal, yet is not married to him at the end, she must have dumped him for someone she likes more, and therefore doesn’t like Bob more than her current partner. If Alice rejected his proposal, she was already with someone she liked more than Bob. ” Wikipedia



For any graph H define $q(H)$ to be the number of odd components of H, i.e., the number of connected components of H consisting of an odd number of vertices.

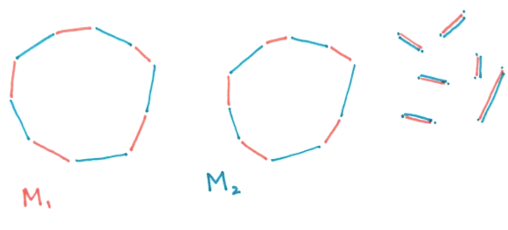
Theorem 2.4 (Tutte's Theorem, 2.2.1)**.** A graph G has a perfect matching if and only if $q(G-S) \leq |S|$ for all $S \subseteq V$.

Proof. Assume first that G has a perfect matching M. Consider a set S of vertices and an odd component G' of $G-S$. We see that there is at least one vertex in G' that is incident to an edge of M that has another endpoint not in G'. This endpoint must be in S. Thus $|S|$ is at least as large as the number of odd components.

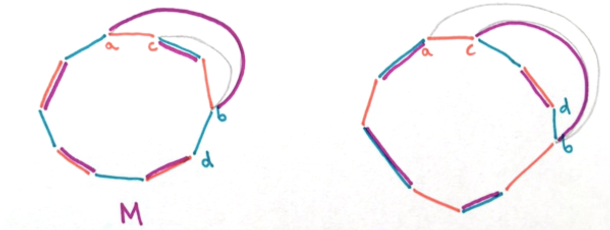


Now, assume that $q(G-S) \leq |S|$ for all $S \subseteq V$. Assume that G has no perfect matching and $|V(G)| = n$. Note that $|V(G)|$ is even (it follows from the assumption $q(G-S) \leq |S|$ applied to $S = \emptyset$). Let G' be constructed from G by adding missing edges as long as no perfect matching appears. Let S be a set of vertices of degree $n-1$. Note that it could be empty.

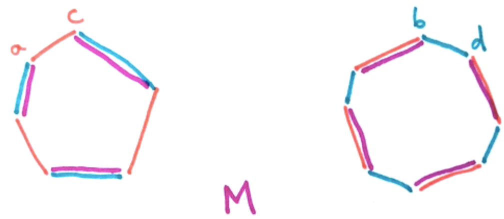
Claim 1. Each component of $G'-S$ is complete. Assume not, there is a component F containing vertices a, b, c such that $ab, bc \in E(G')$ and $ac \notin E(G')$. Since $b \notin S$, $deg(b) < n-1$, so there is $d \in V(G)$, $d \notin \{a, b, c\}$, such that $bd \notin E(G')$. By maximality of G', $G' \cup ac$ has a perfect matching M_1 and $G' \cup bd$ has a perfect matching M_2. Let H be a graph with edge set $E(M_1)\Delta E(M_2)$.



Then H is a vertex-disjoint union of even cycles, alternating edges from M_1 and M_2. We have that $ac, bd \in E(H)$. If ac and bd are in the same cycle C of H, we see that $C \cup \{ab, cb\}$ has a perfect matching M_C not containing either ac or bd. Build a perfect matching M of G' from M_C, a perfect matchings of other components of H, and the edges of $M_1 \cap M_2$.

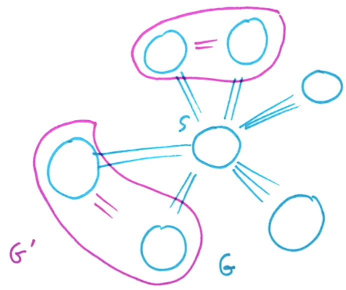


If ac and bd belong to different cycles of H, again build a perfect matching M of G' not containing ac and not containing bd. We see that M is a perfect matching of G', contradicting the assumption that G' has no perfect matching. This proves Claim 1.



Claim 2. $q(G' - S) > |S|$.
If $q(G' - S) \leq |S|$, build a perfect matching of G' by matching a single vertex in each odd component of $G' - S$ to S, matching the remaining vertices in each component of $G' - S$ to the vertices in respective components, and matching the remaining vertices of S to the vertices of S. Since $V(G')$ is even, we can construct a perfect matching in this way, a contradiction. This proves Claim 2.

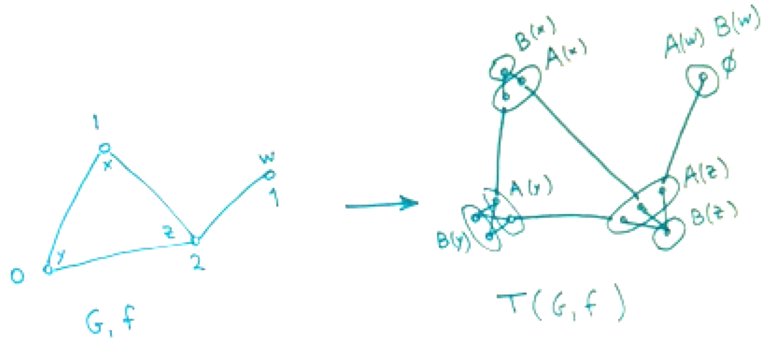
Finally, observe that since G' is obtained from G by adding edges $q(G-S) \geq q(G'-S)$. Thus $q(G-S) > |S|$, a contradiction.



□

Definition 2.5. Let $G = (V, E)$ be any graph.

- For all functions $f\colon V \to \mathbb{N} \cup \{0\}$ an *f-factor of G* is a spanning subgraph H of G such that $\deg_H(v) = f(v)$ for all $v \in V$. f-factor
- Let $f\colon V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \leq \deg(v)$ for all $v \in V$. We can construct the auxiliary graph $T(G, f)$ by replacing each vertex v with vertex sets $A(v) \cup B(v)$ such that $|A(v)| = \deg(v)$ and $|B(v)| = \deg(v) - f(v)$. For adjacent vertices u and v we place an edge between $A(u)$ and $A(v)$ such that the edges between the A-sets are independent. We also insert a complete bipartite graph between $A(v)$ and $B(v)$ for each vertex v. $T(G, f)$



- Let H be a graph. An *H-factor of G* is a spanning subgraph of G that is a vertex-disjoint union of copies of H, i.e., a set of disjoint copies of H in G whose vertex sets form a partition of V. H-factor

Lemma 15. Let $f\colon V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \le \deg(v)$ for all $v \in V$. Then G has an f-factor if and only if $T(G, f)$ has a 1-factor.

Proof. Assume first that G has an f-factor. For each edge uv of the f-factor, consider an edge between $A(u)$ and $A(v)$ such that respective edges form a matching, M. We see that exactly $f(v)$ vertices of $A(v)$ are saturated by M. Build a matching between the unsaturated by M vertices of $A(v)$ and $B(v)$, for each $v \in V(G)$.
Assume that $T(G, f)$ has a perfect matching M. Delete all $B(v)$'s, $v \in V(G)$, and contract $A(v)$ into a single vertex v. After such an operation applied to M, each vertex v has degree $f(v)$ and the graph is clearly a subgraph of G. $\square$

Theorem 16 (Hajnal and Szemerédi 1970)**.** If G satisfies $\delta(G) \ge (1 - 1/k)n$, where k is a divisor of n, then G has a K_k-factor.

Theorem 17 (Alon and Yuster 1995)**.** Let H be a graph. If G satisfies

$$\delta(G) \ge \left(1 - \frac{1}{\chi(H)}\right) n,$$

then G contains at least $\big(1 - o(1)\big) \cdot n/|V(H)|$ vertex-disjoint copies of H.

Theorem 18 (Komlós, Sárközy, and Szemerédi 2001)**.** For any graph H with $\chi(H) = k$, $|H| = r$, there are constants c, n_0 such that for any $n \ge n_0$ such that n is divisible by r and $\delta(G) \ge (1 - 1/k)n + c$, G contains an H-factor.

Theorem 19 (Wang 2010)**.** If $|G| = 4m$ and $\delta(G) \ge n/2$, then G has a C_4-factor.

Definition. For a graph H, define the *critical chromatic number of H* as critical chromatic number, $\chi_{cr}(H)$

$$\chi_{cr}(H) = \frac{(\chi(H) - 1)|H|}{|H| - \sigma(H)},$$

where $\sigma(H)$ denotes the minimum size of the smallest color class in a coloring of H with $\chi(H)$ colors.

Note that for any graph H, $\chi_{cr}(H)$ always satisfies

$$\chi(H) - 1 \le \chi_{cr}(H) \le \chi(H)$$

and $\chi_{cr}(H) = \chi(H)$ if and only if for every coloring of H with $\chi(H)$ colors, all of the color classes have equal size.

Theorem 20 (Kühn and Osthus, 2009)**.** Let H be a graph and $n \in \mathbb{N}$ so that n is divisible by $|H|$ and define

$$\delta(n, H) = \min\{k \ : \ \text{any } G \text{ with } |G| = n,\ \delta(G) \ge k \text{ has an } H\text{-factor}\}.$$

Then there exists a constant $C = C(H)$ so that

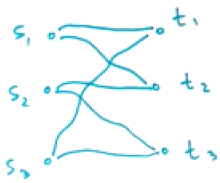
$$\left(1 - \frac{1}{r}\right) n - 1 \le \delta(n, H) \le \left(1 - \frac{1}{r}\right) n + C,$$

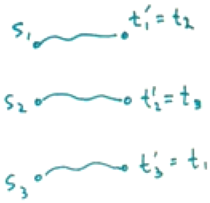
where $r \in \{\chi(H), \chi_{cr}(H)\}$.

3 Connectivity

Definition 3.1.

- For a natural number $k \geq 1$, a graph G is called *k-connected* if $|V(G)| \geq k+1$ and for any set U of $k-1$ vertices in G the graph $G - U$ is connected. In particular, K_n is $(n-1)$-connected.
  k-connected
- The maximum k for which G is k-connected is called the *connectivity of* G, denoted by $\kappa(G)$. For example, $\kappa(C_n) = 2$ and $\kappa(K_{n,m}) = \min\{m, n\}$.
  connectivity, $\kappa(G)$
- For a natural number $k \geq 1$, a graph G is called *k-linked* if $|G| \geq 2k$ and for any $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ there are vertex-disjoint s_i-t_i-paths, $i = 1, \ldots, k$.
  k-linked





- For a graph $G = (V, E)$ a set $X \subseteq V \cup E$ of vertices and edges of G is called a *cut set* of G if $G - X$ has more connected components than G. If a cut set consists of a single vertex v, then v is called a *cut vertex* of G; if it consists of a single edge e, then e is called a *cut edge or bridge* of G.
  cut set
  cut vertex
  cut edge, bridge
- For a natural number $\ell \geq 1$, a graph G is called *ℓ-edge-connected* if G is non-trivial and for any set $F \subseteq E$ of fewer than ℓ edges in G the graph $G - F$ is connected.
  ℓ-edge-connected
- The *edge-connectivity of* G is the maximum ℓ such that G is ℓ-edge-connected. It is denoted by $\kappa'(G)$.
  edge-connectivity, $\kappa'(G)$

$$G \text{ non-trivial tree} \Rightarrow \kappa'(G) = 1,\ G \text{ cycle} \Rightarrow \kappa'(G) = 2.$$

Clearly, for every $k, \ell \geq 2$, if a graph is k-connected, k-linked or ℓ-edge-connected, then it is also $(k-1)$-connected, $(k-1)$-linked or $(\ell-1)$-edge-connected, respectively. Moreover, for a non-trivial graph is it equivalent to be 1-connected, 1-linked, 1-edge-connected, or connected.

Lemma 3.2. For any connected, non-trivial graph G we have

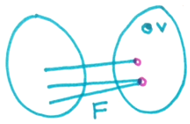
$$\kappa(G) \le \kappa'(G) \le \delta(G).$$

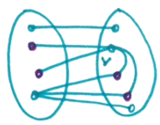
Proof. Observe first that for a complete graph $G = K_n$, $\kappa(G) = \kappa'(G) = \delta(G) = n-1$. So, we can assume that G is not complete.

To show that $\kappa'(G) \le \delta(G)$, observe that G can be disconnected by removing the edges incident to a vertex v of minimum degree. To show that $\kappa(G) \le \kappa'(G)$, consider a smallest separating set of edges, F, of size $\kappa'(G)$. We shall show that $\kappa(G) \le |F|$.

Case 1. There is a vertex v not incident to F. Then v is in the component G' of $G - F$. Then the vertices of G' incident to F separate G, there are at most $|F|$ of them.

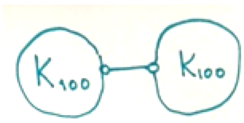
Case 2. Every vertex is incident to F. Let v be a vertex of degree less than $|G| - 1$. Such exists since G is not complete. Let G' be the component of $G - F$ containing v. Then $U = \{u : u \in N(v), uv \notin F\} \subseteq V(G')$. For each $u \in U$, u is incident to F, moreover distinct u's from U are incident to distinct edges from F. So, $|N(v)| \le |F|$. We see that $N(v)$ is a separating set, so $\kappa(G) \le |F|$.



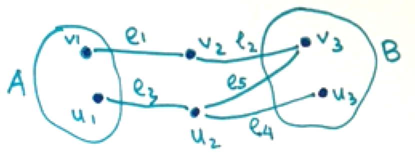


□

Example. A graph G with $\kappa(G), \kappa'(G) \ll \delta(G)$.



Definition. For a subset X of vertices and edges of G and two vertex sets A, B in G we say that X *separates* A *and* B if each A-B-path contains an element of X. separate
Note that if X separates A and B, then necessarily $A \cap B \subseteq X$.



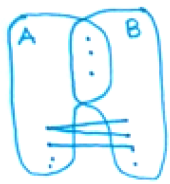
Some sets separating A and B: $\{e_1, e_4, e_5\}$, $\{e_1, u_2\}$, $\{u_1, u_3, v_3\}$

Theorem 3.3 (Menger's Theorem, 3.3.1)**.** For any graph G and any two vertex sets $A, B \subseteq V(G)$, the smallest number of vertices separating A and B is equal to the largest number of disjoint A-B-paths.

Proof. Let $s(A, B)$ be the smallest number of vertices separating A and B, let $p(A, B)$ be the largest number of disjoint A-B-paths. It is clear that $s(A, B) \geq p(A, B)$. To show that $s(A, B) \leq p(A, B)$, we shall prove a stronger statement:

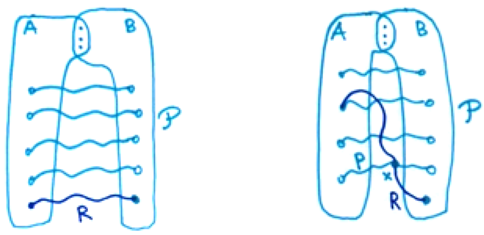
If $\mathcal{P}$ is any set of less than $s(A, B)$ disjoint A-B-paths in G, then there is a set $\mathcal{Q}$ of $|\mathcal{P}| + 1$ disjoint A-B-paths whose set of endpoints includes the set of endpoints of $\mathcal{P}$. We shall fix A and G, vary B, run induction on $|G - B|$.

Basis: $|G - B| = |A - B|$, i.e., there are no vertices outside of $A \cup B$. The result follows from Kőnig's theorem applied to the bipartite subgraph of G with parts $A \backslash B$ and $B \backslash A$.



Step: Assume that $|G - B| = q$ and for all B with $|G - B| < q$ the statement holds.

Let $V(\mathcal{P})$ denote the set of all vertices from $\mathcal{P}$ and let R be an A-B-path that does not contain any vertex from $B \cap V(\mathcal{P})$. Such a path exists, otherwise the set of endpoints of $\mathcal{P}$ in B would separate A and B.

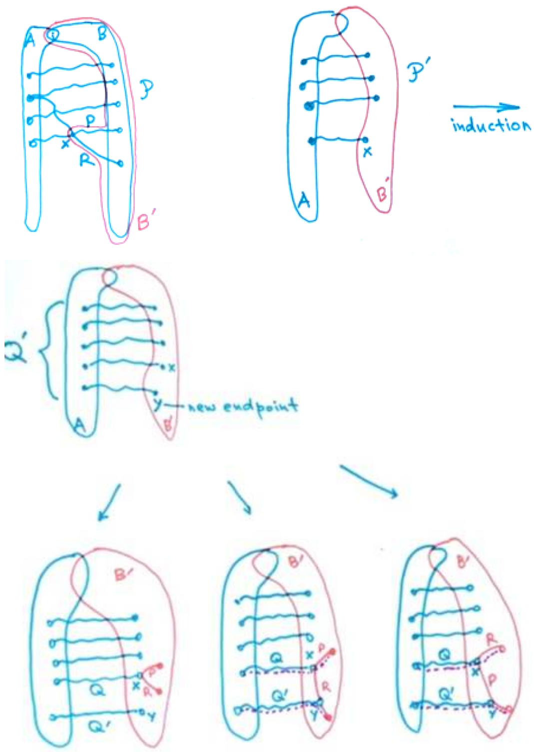


Case 1 $R \cap V(\mathcal{P}) = \emptyset$. Then let $\mathcal{Q} = \mathcal{P} \cup \{R\}$.

Case 2 $R \cap V(\mathcal{P}) \neq \emptyset$. Let $x \in V(\mathcal{P}) \cap V(R)$ such that x is the last such vertex on R. Let $P \in \mathcal{P}$ such that $x \in V(P)$. Let $B' = B \cup V(xP \cup xR)$, let $\mathcal{P}' = \mathcal{P} \setminus \{P\} \cup \{Px\}$. Since $|P'| = |P| < s(A, B) \leq s(A, B')$, by induction there is a set $\mathcal{Q}'$ of $|\mathcal{P}'| + 1$ disjoint A-B'-paths whose set of endpoints contains the set of endpoints of $\mathcal{P}'$. Thus there is $Q \in \mathcal{Q}'$, with endpoint x and there is $Q' \in \mathcal{Q}'$ with endpoint y, where $y \in B'$ and y is not an endpoint of a path in $\mathcal{Q}'$.

Case 1. $y \in B \setminus V(\mathcal{P})$, then let $\mathcal{Q} = \mathcal{Q}' \setminus \{Q\} \cup \{Q \cup xP\}$.
Case 2. $y \in xP$. Let $\mathcal{Q} = (\mathcal{Q}' \setminus \{Q, Q'\}) \cup \{Q' \cup yP, Q \cup xR\}$.
Case 3. $y \in xR$. Let $\mathcal{Q} = (\mathcal{Q}' \setminus \{Q, Q'\}) \cup \{Q' \cup yR, Q \cup xP\}$.

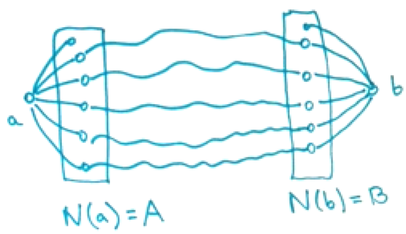
We see that $\mathcal{Q}$ is a desired set of A-B paths.



□

Corollary 21. If a, b are vertices of G, $\{a, b\} \notin E(G)$, then

$$\min \#\text{vertices from } V(G) \setminus \{a, b\} \text{ separating } a \text{ and } b \;=\; \max \#\text{independent } a\text{-}b\text{-paths}$$



Theorem 3.4 (Global Version of Menger's Theorem, 3.3.6)**.** A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Proof. Assume that G is k-connected. Then $|G| > k$ and for any two vertices one needs at least k vertices to separate them. Assume that there are at most $(k-1)$ independent a-b-paths for some distinct vertices a and b.

If a is not adjacent to b, consider $A = N[a]$ and $B = N[b]$. Any A-B-path starts in $N(a)$ and ends in $N(b)$. Thus a set of independent a-b-paths corresponds to a set of disjoint A-B-paths bijectively. Therefore we have at most $(k-1)$ disjoint A-B-paths and thus by Menger's theorem there is a set of at most $k-1$ vertices separating A and B. This set separates a and b. A contradiction.

If a and b are adjacent, consider a graph G' obtained from G by deleting the edge ab. Then there are at most $k-2$ independent a-b-paths in G'. As before, we apply Menger's theorem to $N[a]$ and $N[b]$ in G' and see that there is a set X of at most $k-2$ vertices separating a and b in G'. Since $|G| > k$, there is a vertex $v \notin X \cup \{a, b\}$. Thus X separates v from either a or b, say from a. Then $X \cup \{b\}$ separates v from a in G. Hence the set $X \cup \{b\}$ is a set of $k-1$ vertices separating a and v in G, a contradiction.

Now, assume that there are at least k independent paths between a and b in G, for any two vertices a and b. Thus $|G| > k$ and the deletion of less than k vertices does not disconnect G. $\square$

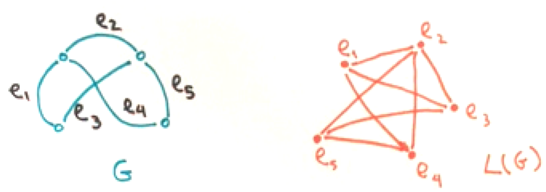
Note that Menger's Theorem implies that if G is k-linked, then G is k-connected.

Theorem 22 (Thomas and Wollan, 2005)**.** If a graph G is $10k$-connected, then it is k-linked.

Definition. For a graph $G = (V, E)$ the *line graph* $L(G)$ of G is the graph $L(G) = (E, E')$, where

line graph $L(G)$

$$E' = \left\{ \{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G \right\}.$$



A graph and its line graph.

Theorem 23 (Beineke, 1970)**.** A graph $\mathcal{L}$ is a line graph of some graph if and only if it does not contain any of the graphs from Figure 1 as induced subgraphs.

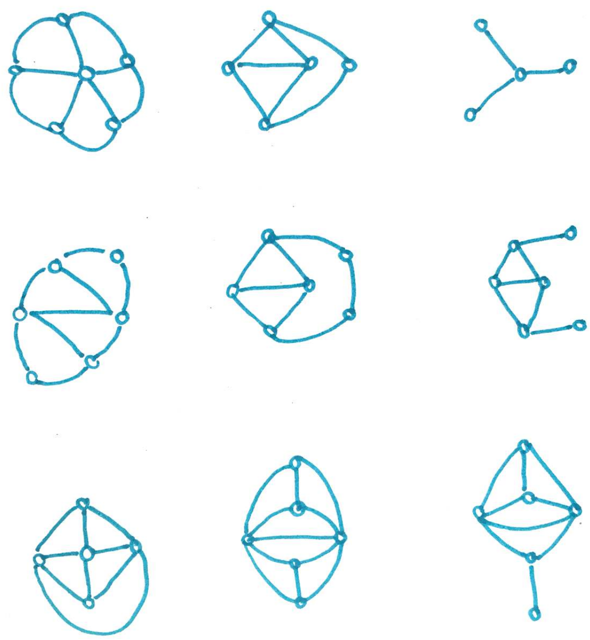
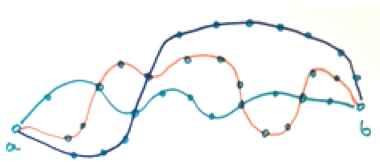


Figure 1: Forbidden induced subgraphs of a line graph.

Corollary 24. If a, b are vertices of G, then

$$\min \#\text{edges separating } a \text{ and } b \;=\; \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$

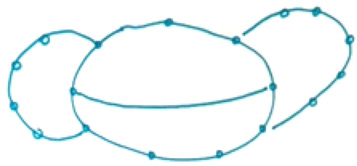


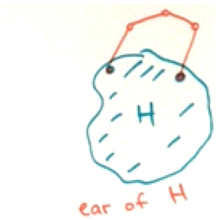
Moreover, a graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices.

Definition 3.5. Given a graph H, we call a path P an *H-path* if P is non-trivial (has length at least one) and meets H exactly in its ends. In particular, the edge of any H-path of length 1 is never an edge of H. We sometimes refer to such a path P as an *ear* of the graph $H \cup P$. H-path

An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$ of graphs, such that ear-decomposition

- G_0 is a cycle,
- for each $i = 1, \ldots, k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e., P_i is an ear of G_i, and
- $G_k = G$.





Theorem 25 (Ear-decomposition)**.** A graph G is 2-connected if and only if it has an ear decomposition starting from any cycle of G.

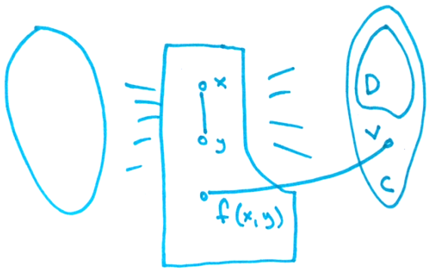
Proof. Assume first that G has a ear-decomposition starting from a cycle C, i.e., $C = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k = G$, where G_i is obtained from G_{i-1} by adding a ear. We shall prove by induction on i that G_i is 2-connected. Clearly G_0 is 2-connected. Assume that G_i is 2-connected. We have that G_{i+1} is obtained from G_i by adding a ear Q. Then G_{i+1} is connected. In addition, if G_{i+1} contains a cut-vertex, it must be on a ear Q. But deleting a vertex from a ear does not disconnect G_{i+1} since a ear is contained in a cycle.

Now assume that G is 2-connected and C is a cycle in G. Let H be the largest subgraph of G obtained by ear decomposition starting with C. We see that H is an induced subgraph of G, otherwise an edge of G with two vertices in $V(H)$ is an ear that could have been added to H. Assume that $H \neq G$. Since G is connected, there is an edge $e = uv$ with $u \in V(H)$ and $v \notin V(H)$. Since $G - u$ is connected, consider a v-w-path P in $G - u$ for some vertex $w \in V(H) - u$. Let w' be the first vertex from $V(H) - u$ on this path. Then $Pw' \cup uv$ is an ear of H, a contradiction to minimality of H.

□

Lemma 26. If G is 3-connected with $G \neq K_4$, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

Proof. Assume not, i.e., for each edge $e = xy$, $G \circ e$ is not 3-connected, i.e., has a 2-cut. This 2-cut must contain the vertex into which x and y were contracted, and some other vertex, which we denote by $f(x, y)$. We see that in G there is a 3-cut, $\{x, y, f(x, y)\}$ for each edge xy. Among all edges of G choose xy to be the one so that deleting $x, y, f(x, y)$ from G creates a smallest component. Let this component be C. Since G has no 2-cut, no proper subset of $\{x, y, f(x, y)\}$ is a cut, so in particular, $f(x, y)$ has a neighbor v in C. Consider a cut $S = \{f(x, y), v, f(f(x, y), v)\}$. Note that since $xy \in E(G)$, x and y are in the same component of $G - S$. Let D be a component of $G - S$ that contains neither x nor y. As before, we see that v has neighbors in D. However, all neighbors of v are in $C \cup \{x, y, f(x, y)\}$. Thus $D \subseteq C - \{v\}$ implying that $|D| < |C|$. A contradiction to minimality of C.



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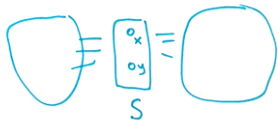
Theorem 3.6 (Tutte, 3.2.3)**.** A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, \ldots, G_k$, such that

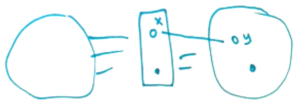
- $G_0 = K_4$,
- for each $i = 1, \ldots, k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x'x''$, and
- $G_k = G$.

Proof. If G is 3-connected, such a sequence exists by Lemma 26. To see that the degree condition is satisfied, recall that $\delta(H) \geq 3$ for any 3-connected graph H. Note that with each contraction, the number of vertices decrease by 1 and until we have at least 5 vertices, we can apply Lemma 26 and contract one more edge. Thus we stop at a graph G_0 which has 4 vertices and $\delta(G_0) \geq 3$ from which $G_0 \cong K_4$ follows.

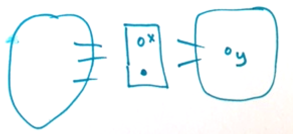
To see the other direction, we shall consider a sequence of graphs satisfying the given conditions and show that each graph in the sequence is 3-connected. Assume that G_i is 3-connected, G_{i+1} is not, and $G_i = G_{i+1} \circ xy$, for an edge xy of G_{i+1} such that $d(x), d(y) \geq 3$. Then G_{i+1} has a cut-set S with at most two vertices.
Case 1. $x, y \in S$.
Then G_i has a cut vertex, a contradiction.



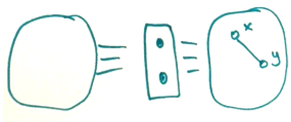
Case 2. $x \in S$, $y \notin S$, y is not the only vertex of its component in $G_{i+1} - S$.
Then G_i has a cut set of size at most 2, a contradiction.



Case 3. $x \in S$, $y \notin S$, y is the only vertex of its component in $G_{i+1} - S$.
Then $d(y) \leq 2$, a contradiction to the fact that $d(y) \geq 3$.



Case 4. $x, y \notin S$.
Then x and y are in the same component of $G_{i+1} - S$. So, S is a cutset of G_i, a contradiction.



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Note that Theorem 3.6 gives a way to generate all 3-connected graphs by starting with K_4 and creating a sequence of graphs by "uncontracting" a vertex such that the degrees of new vertices at at least 3 each.

Theorem 27 (Mader)**.** Every graph $G = (V, E)$ of average degree at least $4k$ has a k-connected subgraph.

Proof. For $k \in \{0, 1\}$ the theorem holds trivially. Let $k \geq 2$. We shall prove a stronger statement $(\star)$ by induction on n, $n = |G|$:

$(\star)$ $|G| \geq 2k-1$ and $||G|| \geq (2k-3)(n-k+1)+1$, then G has a $k-$connected subgraph.

Note that if the assumptions of the theorem hold, i.e., the average degree of G is at least $4k$, then n is at least the maximum degree that is at least the average degree, so $n \geq 4k$ and $||G|| = n4k/2 = 2kn \geq (2k-3)(n-k+1)+1$.

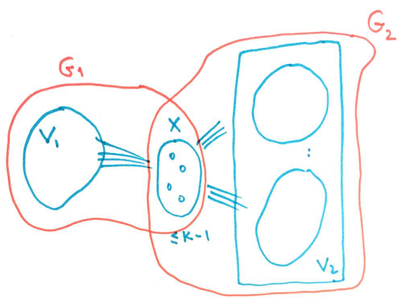
Basis: $n = 2k-1$. Then $k = (n+1)/2$, and $||G|| \geq (2k-3)(n-k+1)+1 = (n-2)(n+1)/2+1 = n(n-1)/2$. Thus G is a complete graph on $2k-1$ vertices, so it is k-connected.

Step: Let $n \geq 2k$ and assume that $(\star)$ holds for smaller values of n.

If v is a vertex of degree at most $2k-3$, apply induction to $G - v$ that has $n-1$ vertices and at least $(2k-3)(n-k+1)+1-(2k-3) = (2k-3)((n-1)-k+1)+1$ edges. By induction $G - v$ has a k-connected subgraph.
Thus we can assume that each vertex has degree at least $2k-2$. If G is not k-connected, then there is a separating set X of vertices, $|X| < k$. Let V_1 be a vertex set of one connected component of $G-X$ and V_2 be vertex sets of all other components of $G-X$. Let $G_i = G[V_i \cup X]$. Each vertex in each V_i has at least $2k-2$ neighbours in G_i, so $|G_1|, |G_2| \geq 2k-1$. Note that $|G_i| < n$, $i = 1, 2$.
If for some $i \in \{1, 2\}$ $||G_i|| \geq (2k-3)(|G_i|-k+1)+1$, then G_i has a k-connected subgraph by induction.
Thus we can assume that $||G_i|| \leq (2k-3)(|G_i|-k+1)$, $i = 1, 2$. Since $|V(G_1) \cap V(G_2)| \leq k-1$, we have that

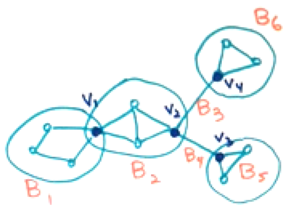
$$||G|| \leq (2k-3)(|G_1|+|G_2|-2k+2) \leq (2k-3)(n-k+1),$$

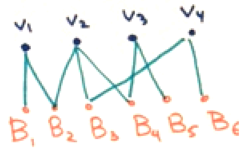
a contradiction. This proves $(\star)$ and the theorem. $\square$



Definition 3.7. Let G be a graph. A maximal connected subgraph of G without a cut vertex is called a *block* of G. In particular, the blocks of G are exactly the bridges and the maximal 2-connected subgraphs of G. block

The *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of G and the cut vertices of G, respectively. There is an edge between a block B and a cut vertex a if and only if $a \in B$, i.e., the block contains the cut vertex. block-cut-vertex graph





The leaves of this graph are called *leaf blocks*. leaf block

Theorem 28. The block-cut-vertex graph of a connected graph is a tree.

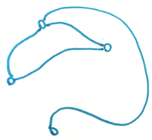
4 Planar graphs

This section deals with graph drawings. We restrict ourselves to graph drawings in the plane $\mathbb{R}^2$. It is also feasible to consider graph drawings in other topological spaces, such as the torus.

Definition 4.1.

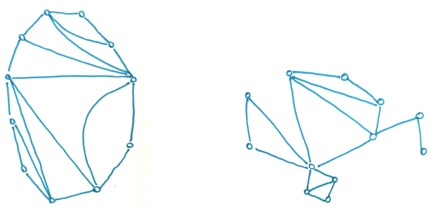
- The *straight line segment* between $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$ is the set $\{p + \lambda(q-p) : 0 \leq \lambda \leq 1\}$. straight line segment
- A *homeomorphism* is a continuous function that has a continuous inverse function. homeomorphism
- Two sets $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}^2$ are said to be *homeomorphic* if there is a homeomorphism $f \colon A \to B$. homeomorphic
- A *polygon* is a union of finitely many line segments that is homeomorphic to the circle $S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$. polygon
- An *arc* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the closed unit interval $[0,1]$. The images of 0 and 1 under such a homeomorphism are the *endpoints of the arc*. If P is an arc with endpoints p and q, then P *links* them and runs *between* them. The set $P \setminus \{p,q\}$ is the *interior of* P, denoted by $\mathring{P}$. arc / endpoint of arc / interior of arc
- Let $O \subseteq \mathbb{R}^2$ be an open set. Being linked by an arc in O is an equivalence relation on O. The corresponding equivalence classes are the *regions of* O. A closed set $X \subseteq \mathbb{R}^2$ is said to *separate* O if $O \setminus X$ has more regions than O. region / separate

  The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \setminus X$. Note that if X is closed, its frontier lies in X, while if X is open, its frontier lies in $\mathbb{R}^2 \setminus X$. frontier
- A *plane graph* is a pair (V,E) of sets with the following properties (the elements of V are again called *vertices*, those in E *edges*): plane graph
  1. $V \subseteq \mathbb{R}^2$;
  2. every $e \in E$ is an arc between two vertices;
  3. different edges have different sets of endpoints;
  4. the interior of an edge contains no vertex and no point of any other edge.

  A plane graph (V,E) defines a graph G on V in a natural way. As long as no confusion can arise, we shall use the name G of this abstract graph also for the plane graph (V,E), or for the point set $V \cup \bigcup E$.

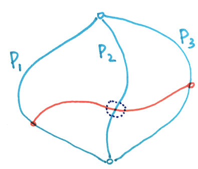
- For any plane graph G, the set $\mathbb{R}^2 \setminus G$ is open; its regions are the *faces of* G. faces, $F(G)$
- The face of G corresponding to the unbounded region is the *outer face* of G; the other faces are its *inner faces*. The set of all faces is denoted by $F(G)$. outer face inner face
- The subgraph of G whose point set is the frontier of a face f is said to *bound* f and is called its *boundary*; we denote it by $G[f]$. boundary of f, $G[f]$
- Let G be a plane graph. If one cannot add an edge to form a plane graph $G' \supsetneq G$ with $V(G') = V(G)$, then G is called *maximally plane*. If every face in $F(G)$ (including the outer face) is bounded by a triangle in G, then G is called a *plane triangulation*. maximally plane triangulation
- A *planar embedding* of an abstract graph $G = (V, E)$ is a bijective mapping $f : V \to V'$, where $G' = (V', E')$ is a plane graph and $uv \in E(G)$, then there is an edge in E' with endpoints $f(u)$ and $f(v)$, We say that G' is a *drawing* of G. We shall not distinguish notational between the vertices of G and G'. A graph $G = (V, E)$ is *planar* if it has a planar embedding. planar embedding planar graph
- A graph $G = (V, E)$ is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all of the vertices V. outerplanar graph



Lemma 29 (Jordan Curve Theorem for Polygons, 4.1.1)**.** Let $P \subseteq \mathbb{R}^2$ be a polygon. Then $\mathbb{R}^2 \setminus P$ has exactly two regions. One of the regions is unbounded, the other is bounded. Each of the two regions has P as frontier.

Lemma 30. Let P_1, P_2 and P_3 be internally disjoint arcs that have the same endpoints. Then

1. $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively.
2. Let P be an arc from the interior of P_1 to the interior of P_3 whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ containing the interior of P_2. Then P contains a points of P_2.



Lemma 31. Let G be a plane graph and e be an edge of G. Then the following hold.

- The frontier X of a face of G either contains e or is disjoint from the interior of e.
- If e is on a cycle in G, then e is on the frontier of exactly two faces.
- If e is on no cycle in G, then e is on the frontier of exactly one face.

Theorem 32 (Plane triangulation)**.** A graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. If G is a plane triangulation, then each face is bounded by a triangle. If an edge is added to G so that the resulting graph is plane, the interior of the the edge must be in some face f of G. The endpoints of the added edge must be two of the three vertices on frontier of f. However, these vertices already are endpoints of an edge of G, a contradiction. Thus no edge could be added to G so that the graph remains plane.

Now assume that G is maximally plane, i.e., that adding any edge violates some property of a plane graph. Let f be a face and $H = G[f]$. Then we see that H is a complete graph, otherwise we could add a new edge with interior in f. If H has at least 4 vertices, $v_1, v_2, v_3, v_4, \ldots$, then we see that v_i-v_j-paths, $i, j \in [4]$ can not all be pairwise disjoint. If H has at most 2 vertices, then f is a face having at most one edge on its boundary, thus $f = \mathbb{R}^2 - G$ and one can add another edge to G. Therefore, we see that H is a complete graph on 3 vertices. □

Theorem 4.2 (Euler's Formula, 4.2.9)**.** Let G be a connected plane graph with n vertices, m edges and ℓ faces. Then

$$n - m + \ell = 2.$$

Proof. We apply induction on m. A connected graph has at least $n-1$ edges. If $m = n-1$, G is a tree. Then $\ell = 1$ and $n-m+\ell = n-(n-1)+1 = 2$. Let $m \geq n$ and assume that the assertion holds for smaller values of m. Then there is an edge e on a cycle. Let $G' = G - e$. Then e is on the boundary of exactly two faces f_1 and f_2. One can show that $F(G') = F(G) - \{f_1, f_2\} \cup \{f'\}$, where $f' = f_1 \cup f_2 \setminus e$. Let n', m', ℓ' be the number of vertices, edges, and faces of G', respectively. Then we see that $n = n'$, $m = m' + 1$, $\ell = \ell' + 1$. So, $n - m + \ell = n - (m' + 1) + (\ell' + 1) = n' - m' + \ell' = 2$ by induction applied to G'. □

Corollary 33. A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation has exactly $3n - 6$ edges.

Proof. We shall prove the second statement. Let m denote the number of edges of G and ℓ denote the number of faces. Each face of a plane triangulation G has exactly three edges on its boundary, every edge is on the boundary of exactly two faces, so $|\{(f, e) : f \in F(G), e \in E(G), e \subseteq \partial f\}| = 3\ell = 2m$. Thus $\ell = 2m/3$. Plugging this into Euler's formula, we obtain $2 = n - m + \ell = n - m + 2m/3 = n - m/3$. Thus $m = 3n - 6$. □

Corollary 34. A triangle-free plane graph with $n \geq 3$ vertices has at most $2n - 4$ edges.

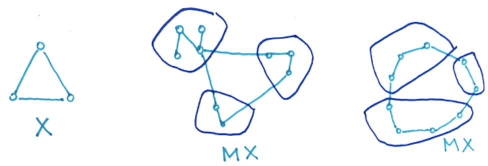
Proof. HW $\square$

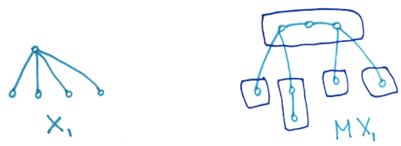
Theorem 35 (Fáry's Theorem)**.** Every planar graph has a plane embedding with straight line segments as edges.

Lemma 36 (Pick's Formula)**.** Let P be a polygon with corners on the grid $\mathbb{Z}^2$, let A be its area, I be the number of grid points strictly inside of P and B be the number of grid points on the boundary of P. Then $A = I + B/2 - 1$.

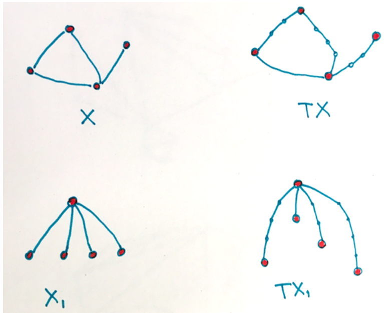
Definition 4.3. Let G and X be two graphs.

- We say that X *is a minor of* G, denoted by $X \preccurlyeq G$, if X can be obtained from G by successive vertex deletions, edge deletions and edge contractions. minor, $X \preccurlyeq G$
- We write $G = MX$ if and only if the vertices G can be partitioned into sets $V(G) = V_1 \dot\cup \cdots \dot\cup V_{|X|}$ such that $G[V_i]$ is connected and for every edge $v_i v_j \in E(X)$, $i, j \in [|X|]$, there is an edge $uv \in E(G)$ incident to the corresponding sets V_i, V_j, i.e., $u \in V_i$ and $v \in V_j$. $G = MX$
  Note that $X \preccurlyeq G$ if and only if $MX \subseteq G$.





- The graph G is a *single-edge subdivision of* X if $V(G) = V(X) \cup \{v\}$ and $E(G) = E(x) - xy + xv + vy$ for $xy \in E(X)$ and $v \notin V(X)$. We say that G is a *subdivision of* X if it can be obtained from X by a series of single-edge subdivisions. subdivision
- We write $G = TX$, if G is a subdivision of X. $G = TX$
- We say that X *is a topological minor of* G, if a subgraph of G is a subdivision of X. Note that X is a topological minor of G if and only if $TX \subseteq G$. topological minor



Theorem 4.4 (Kuratowski's Theorem, 4.4.6)**.** The following statements are equivalent for graphs G:

i) G is planar;

ii) G does not have K_5 or $K_{3,3}$ as minors;

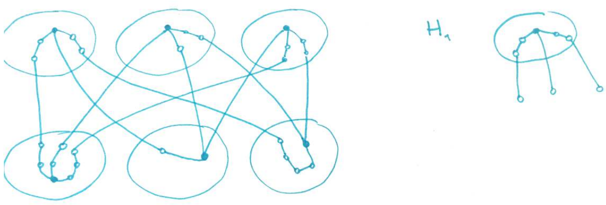
iii) G does not have K_5 or $K_{3,3}$ as topological minors.

Lemmas for the proof of Kuratowski's theorem

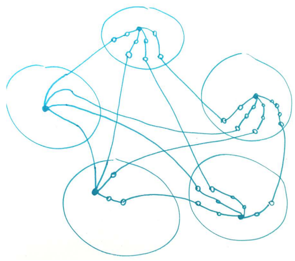
Lemma 37. A graph G contains K_5 or $K_{3,3}$ as a minor iff G contains K_5 or $K_{3,3}$ as a topological minor.

Proof. Assume that $TK_5 \subseteq G$ or $TK_{3,3} \subseteq G$. Since any topological minor is a minor, we have that $MK_5 \subseteq G$ or $MK_{3,3} \subseteq G$. (Note that if we look at TH and MH as classes of graphs, $TH \subseteq MH$).

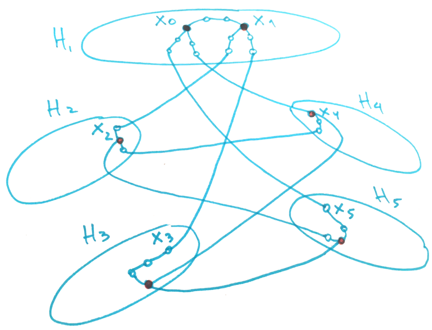
Assume that $MK_{3,3} \subseteq G$. Let H be a smallest (by number of vertices and edges) subgraph of G, such that $H = MK_{3,3}$. There are exactly 9 edges between the branch sets of H, corresponding to 9 edges of $K_{3,3}$. Let H_i be a subgraph of H formed by i^{th} branch set and all the edges of H incident to that branch set, $i = 1, \dots, 6$. By the minimality of H, we see that H_i is a spider with three legs, $i = 1, \dots, 6$, thus $H = TK_{3,3}$.



Finally assume that $MK_5 \subseteq G$. Let H be a smallest (by number of vertices and edges) subgraph of G, such that $H = MK_5$. Then there are exactly 10 edges between the branch sets of H. Let H_i be a subgraph of H formed by i^{th} branch set and all the edges of H incident to that branch set, $i = 1, \dots, 5$. Each H_i is either a 4-legged spider or a tree with exactly two vertices of degree 3 and all other vertices of degrees at most 2. If all H_i's are 4 legged spiders, then $H = MK_5$.



So assume that H_1 is a tree with two vertices, x_0, x_1 of degree 3 and other vertices of degrees at most 2. Without loss of generality, let x_2, x_3 be the vertices from H_1 in 2nd and 3rd branch sets, x_4, x_5 be vertices of H_1 in 4th and 5th branch sets of H, such that there are disjoint x_2-x_1-, x_3-x_1-, x_4-x_0-, and x_5-x_0-paths.



Consider H_2. Then three edges of H_2 that go between 2nd and 1st, 2nd and 4th and 2nd and 5th branch sets are pendant edges of a three-legged spider in H_2. Call its head w_2. Similarly define w_3, w_4, and w_5.

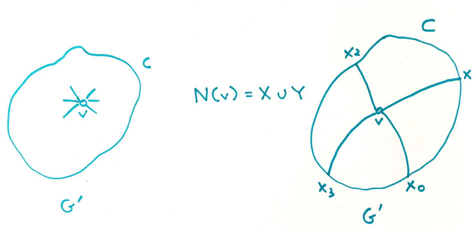
Let y_i, z_i be the vertices in the ith branch set, $i = 2, 3, 4, 5$, such that $y_2y_4, z_2z_5, z_4z_3, y_5y_3$ are edges of H. For each $i = 2, 3, 4, 5$, x_i, y_i, z_i are legs in a three-legged spider, call its head w_i, or endpoints or a path, call some of these endpoints w_i, in the ith branch set. Then we see that H has a $TK_{3,3}$ with branch vertices $\{x_0, w_4, w_5\}$ and $\{x_1, w_2, w_3\}$.

□

Lemma 38. Let G be a 3-connected graph, $MK_5 \not\subseteq G$ and $MK_{3,3} \not\subseteq G$. Then G is planar.

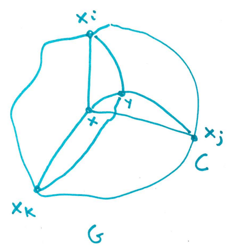
Proof. We shall prove the statement by induction on $|G|$. If $|G| = 4$ we are done as K_4 is the only 3-connected graph on 4 vertices and K_4 is planar. Assume that $|G| > 4$.

Then by Tutte's lemma, there is an edge xy such that $G' = G \circ xy$ is 3-connected. Since G has no K_5 and no $K_{3,3}$ as minors, so does G'. Thus by induction G' is planar. Consider a plane embedding of G'. Let v be a vertex of G' obtained by contracting x and y in G. Let C be a face of $G' - \{v\}$ containing v. Let $X = N_G(x) \setminus \{y\}$ and $Y = N_G(y) \setminus \{x\}$.

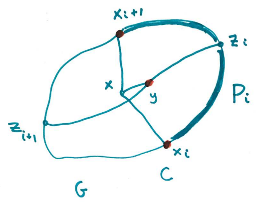


Let $X = \{x_0, \ldots, x_{k-1}\}$ in order on C. Let P_i be an x_i-x_{i+1}-path on C, $i = 0, \ldots, k-1$, addition of indices mod k.

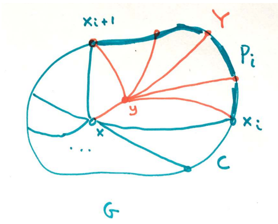
Case 1. $|Y \cap X| \geq 3$. Assume that $x_i, x_j, x_k \subseteq Y \cap X$ for distinct i, j, k. Then x_i, x_j, x_k, x, y form the branch vertices of TK_5 in G, a contradiction.



Case 2. $Y \cap (V(P_i) \setminus \{x_i, x_{i+1}\}) \neq \emptyset$ and $Y \cap (V(C) - V(P_i)) \neq \emptyset$. Let $z_i \in Y \cap (V(P_i) \setminus \{x_i, x_{i+1}\})$ and $z_i' \in Y \cap (V(C) - V(P_i))$. Then $\{y, x_i, x_{i+1}\} \cup \{x, z_i, z_{i+1}\}$ are branch sets of $TK_{3,3}$ in G, a contradiction.



Case 3. $Y \subseteq V(P_i)$ for some i. Embed x as v and y in the region bounded by vx_i, P_i, vx_{i+1}.



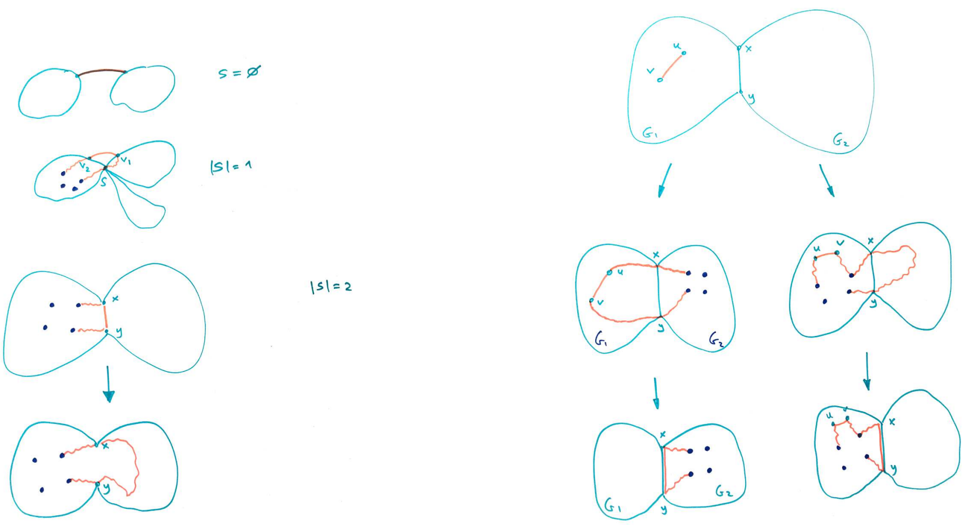
∎

Lemma 39. Let X be a 3-connected graph, G be edge-maximal with respect to not containing TX. Let S be a vertex cut of G, $|S| \leq 2$. Then $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = S$, G_i is edge-maximal without TX and S induces an edge.

Proof. If $|S| = 0$, add an edge between two components.

If $S = \{v\}$, let $v_i \in N(v) \cap V(G_i)$, $i = 1, 2$. Consider $H = TX$ in $G + v_1v_2$. All branch vertices of H must be in either G_1 or G_2, assume without loss of generality in G_1. Then we see that G_1 contains TX by replacing a path of H with vv_1 if needed.

If $S = \{x, y\}$, assume that x and y are not adjacent. Consider $H = TX$ In $G + xy$. Again, all branch vertices of H are without loss of generality in G_1. Moreover $xy \in E(H)$. Replace xy with a path in G_2 to obtain a copy of TX in G. This a contradiction, so xy is an edge in G. To show that G_1 is edge-maximal with respect to not containing TX, consider $H = TX$ in $G + uv$, $u, v \in V(G_1)$. All branch vertices of H is either in G_2 or in G_1. If they all are in G_2, replace a path of H through uv with one through xy. This results in TX in G_2, a contradiction. Thus, all branch vertices of H are in G_1. Replacing a path of H that is in G_2 with xy if needed, we see that $TX \subseteq G_1 + \{uv\}$. This shows the maximality of G_1 with respect to not containing TX.



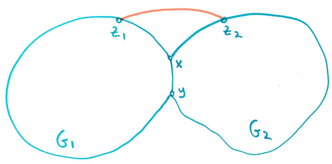
□

Lemma 40. Let $|G| \geq 4$ and G is edge maximal with respect to not containing TK_5 or $TK_{3,3}$. Then G is 3-connected.

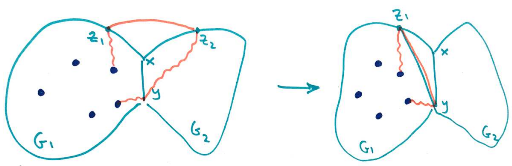
Proof. We use induction on $|G|$. We are done if $|G| = 4$. Assume that $|G| > 4$. Assume that G satisfies the conditions of the lemma but is not 3-connected, i.e., it contains a vertex cut $S = \{x, y\}$. Let $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = S$.

We have that $TK_5 \not\subseteq G_i$, $TK_{3,3} \not\subseteq G_i$, and $|G_i| < |G|$, so by induction G_i is 3-connected, $i = 1, 2$. Since $TK_5 \not\subseteq G_i$ and $TK_{3,3} \not\subseteq G_i$, $i = 1, 2$, we have by Lemma 37 that $MK_5 \not\subseteq G_i$ and $MK_{3,3} \not\subseteq G_i$, $i = 1, 2$. Since $MK_5 \not\subseteq G_i$ and $MK_{3,3} \not\subseteq G_i$, $i = 1, 2$ and G_i's are 3-connected or K_3, we have by Lemma 38 G_i's are planar. We have that $xy \in E(G)$ by Lemma 39.

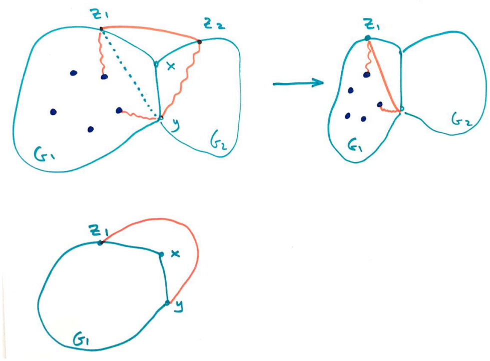
Consider embeddings of G_1 and G_2 so that xy is on the boundary of unbounded face. Let $z_i \in G_i$, $i = 1, 2$, $z_i \notin \{x, y\}$ on the boundary of the respective unbounded face. Then $G + \{z_1 z_2\}$ contains a subgraph H that is TK_5 or $TK_{3,3}$.



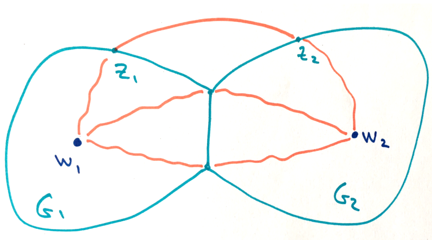
Case 1. The branch vertices of H are in G_i, say $i = 1$. If $xz_1, yz_1 \in E(G_1)$, G_1 contains TK_5 or $TK_{3,3}$.



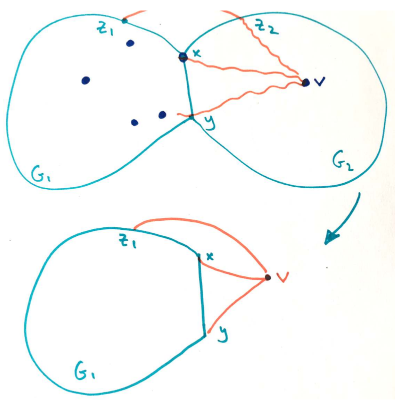
If $xz_1 \notin E(G_1)$ then $G_1 + xz_1$ contains TK_5 or $TK_{3,3}$. If $yz_1 \notin E(G_1)$ then $G_1 + yz_1$ contains TK_5 or $TK_{3,3}$. However, $G_1 + xz_1, G_1 + yz_1$, and G_1 are planar, a contradiction.



Case 2. There are branch vertices of H in $G_1 \setminus G_2$ and in $G_2 \setminus G_1$. Let W_i be the set of branch vertices of H in $V(G_i)$, $i = 1, 2$.



Since there are at most 2 independent paths between $w_i \in W_i$ and $w_j \in W_j$, we see that $H \neq TK_5$. So, $H = TK_{3,3}$. We see that either $|W_1 \cap V(G_1 - G_2)| = 1$ or $|W_2 \cap V(G_2 - G_1)| = 1$. Assume that $|W_2 \cap V(G_2 - G_1)| = 1$ and let $v = W_2 \cap V(G_2 - G_1)$. Then $G' = G_1 + \{v\} + \{vx, vy, vz_1\}$ contains $TK_{3,3}$. But G' is planar, a contradiction.



□

Proof of Kuratowski's theorem

Theorem 4.5 (Kuratowski's Theorem)**.** The following statements are equivalent for graphs G:

i) G is planar;

ii) G does not have K_5 or $K_{3,3}$ as minors;

iii) G does not have K_5 or $K_{3,3}$ as topological minors.

Proof. The equivalence of ii) and iii) follows from Lemma 37.

Assume i). Note that K_5 is not planar since $10 = ||K_5|| > 3|K_5| - 6 = 9$, violating the Euler's formula. Additionally $K_{3,3}$ is not planar since $9 = ||K_{3,3}|| > 2|K_{3,3}| - 4 = 8$, violating the Euler's formula for triangle-free graphs. Since K_5 and $K_{3,3}$ are not planar, TK_5 and $TK_{3,3}$ are not planar, otherwise one can create and embedding of K_5 from one of TK_5 by "merging" the edges resulted from subdivisions. Similarly, $TK_{3,3}$ is not planar, so G does not contain TK_5 and G does not contain $TK_{3,3}$. This implies iii).

We only need to show that ii) implies i). Let G be a graph that contains neither MK_5 nor $MK_{3,3}$. Add as many edges as possible to preserve this property, let the resulting graph be G'. By Lemmas 37 and 40, G' is 3-connected. Lemma 38 states that G' is planar. Thus a subgraph G of G' is planar. This implies i).

$\square$

Definition 4.6.

- Let X be a set and $\leq \subseteq X^2$ be a relation on X, i.e., $\leq$ is a subset of all ordered pairs of elements in X. Then $\leq$ is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if $x \leq y$ or $y \leq x$ for every $x, y \in X$. partial order total order

- Let $\leq$ be a partial order on a set X. The pair $(X, \leq)$ is called a *poset* (partially ordered set). If $\leq$ is clear from context, the set X itself is called a poset. The *poset dimension of* $(X, \leq)$ is the smallest number d such that there are total orders $R_1, \ldots, R_d$ on X with $\leq = R_1 \cap \cdots \cap R_d$. poset poset dimension, $\dim(X, \leq)$

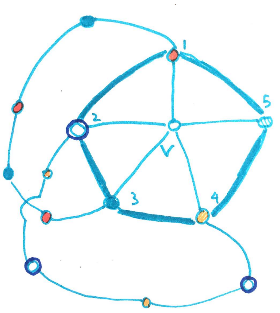
$$\dim(\,\vdots\,) = 1,\ \dim(\underset{x}{\bullet}\ \ \underset{y}{\bullet}) = 2 \text{ since } \underset{x}{\bullet}\ \ \underset{y}{\bullet} = \overset{x}{\underset{y}{\vdots}} \cap \overset{y}{\underset{x}{\vdots}}$$

- The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e$ if and only if e is incident to v for all $v \in V$ and $e \in E$. incidence poset

Theorem 41 (Schnyder)**.** Let G be a graph and P be its incidence poset. Then G is planar if and only if $\dim(P) \leq 3$.

Theorem 4.7 (5-Color Theorem, 5.1.2)**.** Every planar graph is 5-colorable.

Proof. We shall apply induction on $|V(G)|$ with a trivial basis when $|V(G)| \leq 5$. Assume that $|V(G)| > 5$, assume further that G is maximally planar, i.e., it has a plane embedding that is a triangulation. By Euler's formula, there is a vertex v of degree at most 5. By induction, there is a proper coloring c of $G - v$ in at most 5 colors from $[5]$. If c assigns at most 4 colors to $N(v)$, we can assign v a color from $[5]$ not used in $N(v)$. Otherwise, assume w.l.o.g. that $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$, $c(v_i) = i$, and v_i's are cyclically arranged on the face of $G - v$. Let c' be a coloring obtained by $1, 3$ switch at v_1. If $c'(v_1) = 3$ and $c'(v_3) = 3$, then c' does not use color 1 on $N(v)$ and we can color v with 1. So, there is a v_1-v_3-path colored 1 and 3 in c. Similarly, there is a v_2-v_4-path colored 2 and 4 in c. However, this is impossible since these paths must cross is a vertex and this vertex should have a color in $\{1, 3\} \cap \{2, 4\}$.



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The more well-known 4-coloring theorem is much harder to prove.

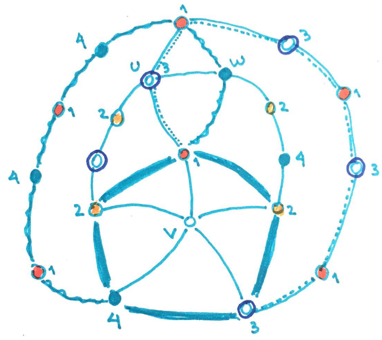
Wrong proof of Four Color Theorem by Kempe

Consider a graph G and a proper vertex coloring c using colors from $[k]$. For a vertex v of color i, we say that a coloring c' is obtained from c by an i, j *color switch at* v if the colors i and j are switched in the maximal connected subgraph of G that is induced by vertices of color i and j and contain v.

The idea of Kempe was to prove a Four-Color theorem using color switches and induction on the number of vertices as follows. Consider a planar graph on n vertices. If $n \leq 4$, the graph is four-colorable. Assume that $n > 4$ and that each planar graph on less than n vertices is 4-colorable. By Euler's formula there is a vertex, v, of degree at most 5. Let c be a proper coloring of $G' = G - v$ with at most four colors from $[4]$. If the number of colors used on $N(v)$ is at most 3, we see that v could be colored with a color from $[4]$ not used on its neighbors. Thus we can assume that the degree of v is 4 or 5 and all four colors are present on $N(v)$. If degree of v is 4, assume that the colors

$1, 2, 3, 4$ appear cyclically on $N(v)$ on vertices v_1, v_2, v_3, v_4 respectively. First, apply $1, 3$ color switch at v_1. If during this switch the color of v_3 remain 3, we can color v with color 1. Thus, there is a v_1-v_3-path colored only with 1 and 3. Similarly there is a v_2-v_4-path colored with 2 and 4 only. However these paths cross, a contradiction. Therefore one of these 1-3 or 2-4 switches results in the neighborhood of v having only three colors and thus v could be colored with the fourth color. Now, assume that degree of v is 5 and the neighbors are colored $1, 2, 3, 4, 2$, on vertices $v_1, \ldots, v_5$ respectively. As before we can assume that there is a v_1-v_3-path colored with 1 and 3 and a v_1-v_4-path colored with 1 and 4. Then, we could do 2-3-switch at v_4 and 2-4-switch at v_2 that results in $N(v)$ loosing color 2. Thus we can color v with 2.

However, there is a problem, see figure. The 2-3 and 2-4 switches resulted in two adjacent vertices u and w of color 2. This mistake was found in 1890 by P. Heawood, after the conjecture was published by De Morgan in 1860, and "proved" by A. Kempe in 1878 (published in Nature).



Interestingly, it is one of the first theorems that has been proved using computer assistance. The computer-generated proof uses an enormous case distinction. Some mathematicians have philosophical problems with this approach since the resulting proof cannot be easily verified by humans. A shorter proof is still outstanding.

Theorem 4.8 (Appel and Haken, 1976)**.** Every planar graph is 4-colorable.

Definition 4.9.

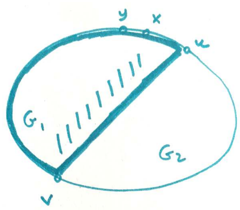
- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is coloring $c\colon V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors. L-list-colorable
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable or k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$. k-list-colorable
- The *choosability*, denoted by $\mathrm{ch}(G)$, is the smallest k such that G is k-choosable. choosability, $\mathrm{ch}(G)$
- The *edge choosability*, denoted by $\mathrm{ch}'(G)$, is defined analogously. edge choosability, $\mathrm{ch}'(G)$

We say that a plane graph is *outer triangulation* if it has all triangular inner faces and an outer face forming a cycle.

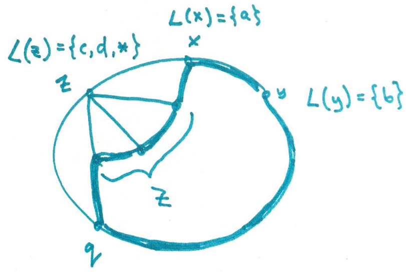
Theorem 4.10 (5-List-Color Theorem)**.** Let G be a planar graph. Then the list chromatic number of G is at most 5.

Proof. We shall prove a stronger statement ($\star$): Let G be an outer triangulation with two adjacent vertices x, y on the boundary of the outer face. Let $L : V(G) \to 2^{\mathbb{N}}$ be a list assignment such that $|L(x)| = |L(y)| = 1$, $L(x) \neq L(y)$, $|L(z)| = 3$ for all other vertices on unbounded face, and $|L(z)| = 5$ for all vertices not on unbounded face. Then G is L-colorable.

We shall prove ($\star$) by induction on $|V(G)|$ with an obvious basis for $|V(G)| = 3$. Consider an outer triangulation G on more than 3 vertices.
Case 1. There is a chord, i.e., an edge uv joining two non-consecutive vertices of the outer face. Then $G = G_1 \cup G_2$, such that $\{u, v\} = V(G_1) \cap V(G_2)$, $|G| > |G_i| \geq 3$, G_i is an outer triangulation, $i = 1, 2$. Without loss of generality, x, y are on the outer face of G_1. Apply induction to G_1 to obtain a proper L-coloring c' of G_1. Next apply induction to G_2 with u and v playing a role of x and y and list assignments L' such that $L'(u) = \{c'(u)\}$, $L'(v) = \{c'(v)\}$, $L'(z) = L(z)$, for $z \notin \{x, y\}$. Then there is a proper L'-coloring c'' of G_2. Since these colorings coincide on u and v, together they form a proper coloring c of G, i.e., $c(v) = c'(v)$ for $v \in V(G_1)$ and $c(v) = c''(v)$ for $v \in V(G_2)$.



Case 2. There is no chord, i.e. Case 1. does not hold. Let z be a neighbor of x on the boundary of outer face, $z \neq y$. Let Z be the set of neighbors of z not on the outer face. Let $L(x) = \{a\}$, $L(y) = \{b\}$. Let $c, d \in L(z)$ such that $c \neq a$ and $d \neq a$. Let $G' = G - z$. Let L' be list assignment for $V(G')$ such that $L'(v) = L(v) - \{c, d\}$, for $v \in Z$ and $L'(v) = L(v)$ for $v \notin Z$. By induction G' has a proper L'-coloring c'. We shall extend a coloring c' to a coloring c or G, i.e., we let $c(v) = c'(v)$ if $v \neq z$. We shall give z a color c or d. Specifically, let $c(z) \in \{c, d\} \setminus \{c'(q)\}$, where q is the neighbor of z on outer face, not equal to x. We see then that z has a color different from the color of each of its neighbors. Thus c is a proper L-coloring.



$\square$

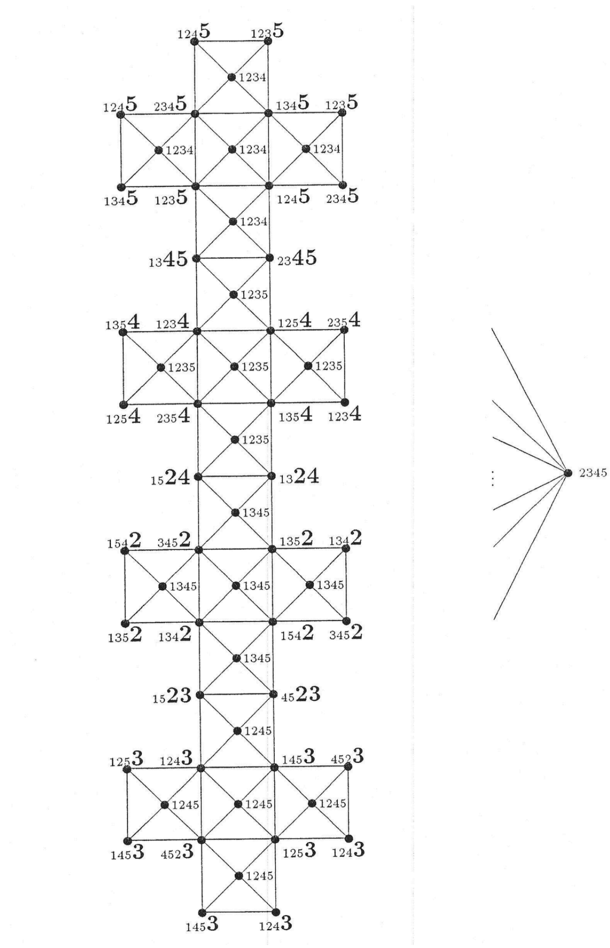


Figure 2: A construction by Mizrakhani of a non-4-choosable planar graph.

An embedding of a graph on a surface is 2-*cell* if for each region and each simple closed curve in the region contracts continuously to a point. Euler formula states that $n - e + f = 2 - 2\gamma$, where $n - e + f$ is called Euler's characteristic, and 2γ is Euler's genus. For orientable surfaces γ corresponds to the number of handles.

Theorem 42 (Heawood formula, 1890 (Weisstein, Ringel 1968, 1974))**.** Let G be embeddable to a surface S with Euler characteristic $2 - 2\gamma$, $\gamma > 0$. Then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor = f(\gamma).$$

Moreover, a complete graph $K_{f(\gamma)}$ is embeddable on S, unless S is a Klein bottle. For a Klein bottle $f(\gamma) = 7$, however, $\chi(G) \leq 6$ and K_6 is embeddable on the Klein bottle.

5 Colorings

Lemma 43. For any connected graph G and for any vertex v there is an ordering of the vertices of G: $v_1, \dots, v_n$ such that $v = v_n$ and for each i, $1 \le i < n$, v_i has a higher indexed neighbor.

Proof. Consider a spanning tree T of G and create a sequence of sets $X_1, \dots, X_{n-1}$ with $X_1 = V(G_1)$, $X_i = X_{i-1} - \{v_{i-1}\}$, where v_i is a leaf of $T[X_i]$ not equal to v, for $i = 1, \dots, n-1$. Then $v_1, \dots, v_n$ is a desired ordering with $v_n = v$. $\square$

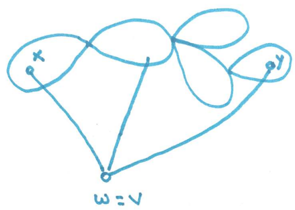
Corollary 44 (Greedy estimate for the chromatic number)**.**
Let G be a graph. Then $\chi(G) \le \Delta(G) + 1$.

Lemma 45. Let G be a 2-connected non-complete graph of minimum degree at least three. Then there are vertices x, y, and v such that $xy \notin E(G)$, $xv, yv \in E(G)$, and $G - \{x, y\}$ is connected.

Proof. Consider a vertex w of degree at most $|G| - 2$.

Case 1. $G - w$ has no cutvertices. Let $x = w$, y be a vertex at distance 2 from x and v be a common neighbor of x and y. Since y is not a cut-vertex in $G - x$, $G - \{x, y\}$ is connected.

Case 2. $G - w$ has a cutvertex. In this case, let $v = w$. Then v must be adjacent to non-cutvertex members of each leaf-block of $G - v$. Let x and y be such neighbors in distinct leaf-blocks. Since v has another neighbor besides x and y, $G - \{x, y\}$ is connected.



$\square$

Theorem 5.1 (Brook's Theorem, 5.2.4)**.** Let G be a connected graph.
Then $\chi(G) \le \Delta(G)$ unless G is a complete graph or an odd cycle.

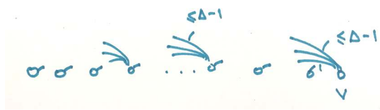
Proof. We shall prove the result by induction on n. The theorem holds for any graph on at least three vertices. Assume that $|G| > 3$.

If G has a cut-vertex v, we can apply induction to the graphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$ and $|G_1| < |G|$ and $|G_2| < G$. Indeed, if each of G_1 and G_2 is not complete or an odd cycle, then $\chi(G_i) \le \Delta(G_i) \le \Delta(G)$,

$i = 1, 2$. If G_i is a complete graph or an odd cycle for some i, $\Delta(G_i) < \Delta(G)$ and $\chi(G_i) = \Delta(G_i) + 1 \leq \Delta(G)$. By making sure that the color of v is the same in an optimal proper coloring of G_1 and G_2 we see that $\chi(G) \leq \Delta(G)$.

Note also that if $\Delta(G) \leq 2$, the theorem holds trivially. So, we assume that $\Delta(G) \geq 3$. So, we can assume that G is 2-connected. We shall show that G can be properly colored with colors from $\{1, \ldots, \Delta\}$, where $\Delta = \Delta(G)$.

Case 1 There is a vertex v of degree at most $\Delta - 1$. We shall order the vertices of G $v_1, \ldots, v_n$ such that $v = v_n$ and each v_i, $i < n$ has a neighbor with a larger index. Such an ordering exists by Lemma 1. Color G greedily with respect to this ordering. We see at step i, there are at most $\Delta - 1$ neighbors of v_i that has been colored, so there is an available color for v_i.



Case 2 All vertices of G have degree Δ. Consider vertices x, y, v guaranteed by Lemma 45, i.e., such that $xy \notin E(G)$ and $xv, yv \in E(G)$, and $G - \{x, y\}$ is connected. Order the vertices of G as $v_1, \ldots, v_n$ such that $v_1 = x$, $v_2 = y$, $v_n = v$ and for each v_i, $3 \leq i < n$ there is a neighbor of v_i with a higher index, such an ordering exists by Lemma 43. Color G greedily according to this ordering. We see that v_1 and v_2 get the same color and as in the previous case, at step i, $3 \leq i < n$, v_i has at most $\Delta - 1$ colored neighbors so it could be colored with a remaining color. At the last step, we see that v_n has Δ colored neighbors, but two of them, v_1 and v_2 have the same color, so there are at most $\Delta - 1$ colors used by the neighbors of v_n. Thus v_n can be colored with a remaining color.

□

Definition 5.2.

- The *clique number* $\omega(G)$ *of* G is the largest order of a clique in G. clique number, $\omega(G)$
- The *co-clique number* $\alpha(G)$ *of* G is the largest order of an independent set in G. co-clique number, $\alpha(G)$
- A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. perfect graph
  For example, bipartite graphs are perfect with $\chi = \omega = 2$.

Lemma 46 (Simple Coloring Results)**.** For any graph G the following hold:

- $\chi(G) \geq \max\{\omega(G), |G|/\alpha(G)\}$,
- $\|G\| \geq \binom{\chi(G)}{2}$ and $\chi(G) \leq 1/2 + \sqrt{2\|G\| + 1/4}$,
- $\chi(G)$ of G is at most one more than the length of a longest directed path in any orientation of G. Moreover, equality holds for some orientation of G.

Proof. The first item holds since $\chi(G) \geq \chi(K_\omega) = \omega$ and each color class in a proper vertex-coloring is an independent set.
The second item holds since in a proper coloring with $\chi(G)$ colors there is an edge between any two color classes (otherwise one can replace these two color classes with their union as a new color class).
To prove the last item, consider an arbitrary orientation D of G. Let D' be a maximal subdigraph of D that contains no oriented cycle. Note that D' is spanning. For all $v \in V(G)$, let $c(v)$ be equal to the length of a longest directed path that ends at v (if there is no such path, we set $c(v) = 0$). Let P be a path in D' that starts at u. Since D' is acyclic, every path in D' that end at u has no other vertex on D'. Thus any path ending at u can be lengthened along P. This implies that c strictly increases along each path of D'. We claim that c is a proper coloring. For each edge $uv \in E(G)$, there is a directed path in D' between its endpoints (either uv is an edge of D' or its addition to D' creates a directed cycle). It implies that $c(u) \neq c(v)$, since c strictly increases along each path in D'. On the other hand, we can create an orientation of G such that a longest directed path has length at most $\chi(G) - 1$ by coloring the vertices of G with the colors $\{1, 2, \ldots, \chi(G)\}$ and orienting each edge from smaller to larger color class. $\square$

Theorem 47 (Lovász' Perfect Graph Theorem, 5.5.4)**.** A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Theorem 48 (Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour & Thomas, 5.5.3)**.** A graph G is perfect if and only if it does not contain an odd cycle on at least 5 vertices (an *odd hole*) or the complement of an odd hole as an induced subgraph.

Theorem 49 (Spectral Theorem)**.** Let A be the adjacency matrix of a graph G. Then A is a symmetric matrix, has an orthonormal basis of eigenvectors and all of its eigenvalues are real.

Definition 5.3. Let A be the adjacency matrix of a graph G.

- The *spectrum* $\lambda(G)$ *of* G is the multiset of eigenvalues of A. spectrum, $\lambda(G)$
- The *spectral radius of* G is $\lambda_{\max}(G) := \max\{\lambda : \lambda \in \lambda(G)\}$. spectral radius, $\lambda_{\max}(G)$
  Analogously, $\lambda_{\min}(G) := \min\{\lambda : \lambda \in \lambda(G)\}$.

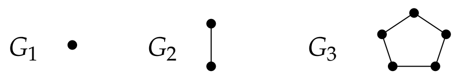
Lemma 50 (Small results about the eigenvalues of G). Let A be the adjacency matrix of G and let H be an induced subgraph of G. Then

- $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$,
- $\delta(G) \leq 2\|G\|/n \leq \lambda_{\max}(G) \leq \Delta(G)$,
- $\mathrm{trace}(A) = 0$, $\mathrm{trace}(A^2) = 2\|G\|$, $\mathrm{trace}(A^3) = 6 \cdot \#$ triangles in G.

Theorem 51 (Spectral estimate for the chromatic number).
Let G be a graph. Then $\chi(G) \leq \lambda_{\max}(G) + 1$.

Example (Mycielski's Construction).
We can construct a family $\big(G_k = (V_k, E_k)\big)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows:

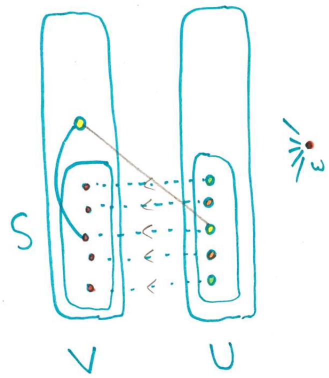
- G_1 is the single-vertex graph, G_2 is the single-edge graph, i.e., $G_1 = K_1$ and $G_2 = K_2$.
- $V_{k+1} := V_k \cup U \cup \{w\}$ where $V_k \cap \big(U \cup \{w\}\big) = \emptyset$, $V_k = \{v_1, \dots, v_n\}$ and $U = \{u_1, \dots, u_n\}$.
- $E_{k+1} := E_k \cup \big\{wu_i : i = 1, \dots, k\big\} \cup \bigcup_{i=1}^{n} \big\{u_i v : v \in N_{G_k}(v_i)\big\}$.



Lemma 52. For any $k \geq 1$, Mycielski's graph G_k has chromatic number k. Moreover, G_k is triangle-free.

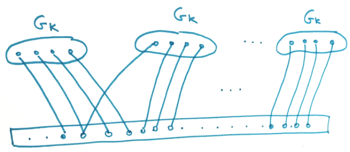
Proof. We shall prove this statement by induction on k with trivial basis $k = 1$. Assume that $k \geq 2$ and $\chi(G_{k-1}) = k-1$ and G_{k-1} is triangle-free. First we show that $\chi(G_k) = k$. We see that $\chi(G_k) \leq k$ by considering a proper coloring c of G_{k-1} with colors from $[k-1]$, and letting $c' : V_k \to [k]$ such that $c'(v_i) = c(v_i)$ for $v_i \in V_{k-1}$, $c'(u_i) = c'(v_i)$, $u_i \in U_k$, $c'(w) = k$. Since $N(u_i) - \{w\} = N(v_i) \cap V_{k-1}$, the coloring is proper and so $\chi(G_k) \leq k$.

Now assume that $\chi(G_k) < k$. Let c be a proper coloring of G_k with colors from $[k-1]$. We know that since $G_{k-1} \subseteq G_k$, and $\chi(G_{k-1}) = k-1$, that all colors from $[k-1]$ are used in c. Assume without loss of generality that $c(w) = k-1$. Then all vertices in U_{k-1} are colored from $[k-2]$. We shall show that the vertices in V_{k-1} could also be colored from $[k-2]$. Let $S \subseteq V_{k-1}$ be the set of vertices in V_{k-1} of color $\{k-1\}$. Recolor $v_i \in S$ with $c(u_i)$, for each $v_i \in S$. We claim that the resulting coloring of G_{k-1} is proper. Assume not and some $v_i \in S$ is adjacent to a vertex x of color $c(u_i)$. Since S is an independent set $x \notin S$. If $x = v_j \notin S$, then u_i is adjacent to v_j, so $c(u_i) \neq c(v_j)$, a contradiction.



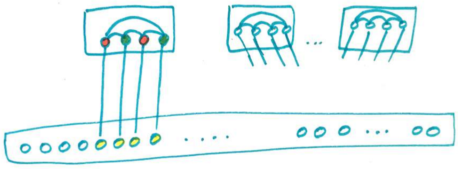
To see that G_k has no triangles, observe that a triangle could only have one vertex in $u_i \in U_{k-1}$ and two vertices in $v_j, v_m \in V_{k-1}$. Then v_i, v_j, v_m form a triangle in G_{k-1}, a contradiction. $\square$

Example (Tutte's Construction)**.** We can construct a family $(G_k)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows: G_1 is the single-vertex graph. To get from G_k to G_{k+1}, take an independent set U of size $k(|G_k| - 1) + 1$ and $\binom{|U|}{|G_k|}$ vertex-disjoint copies of G_k. For each subset of size $|G_k|$ in U then introduce a perfect matching to exactly one of the copies of G_k.



Lemma 53. For any k, Tutte's graph G_k has chromatic number k and it is triangle-free.

Proof. We argue by induction on k with trivial basis $k = 1$. We see that $\chi(G_k) \leq \chi(G_{k-1}) + 1$ because we can assign the same set of $\chi(G_{k-1})$ colors to each copy of G_{k-1} and a new color to U. Assume that $\chi(G_k) \leq \chi(G_{k-1})$. Consider a coloring of G_k with $\chi(G_{k-1})$ colors. By pigeonhole principle there is a set U' of $|G_{k-1}|$ vertices in U of the same color, say 1. The vertices of U' are matched to a copy G' of G_{k-1}. Then G' does not use color 1 on its vertices and thus colored with less than $\chi(G_{k-1})$ colors. Therefore there are two adjacent vertices of the same color. So, any proper coloring of G_k uses more than $\chi(G_{k-1})$ colors.



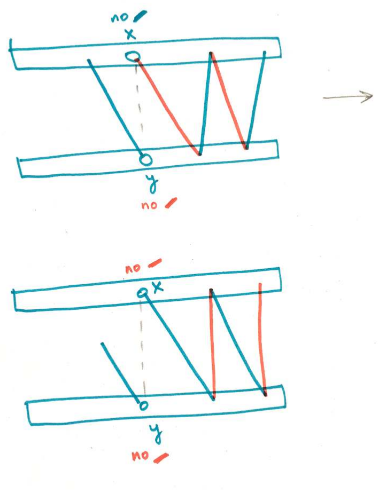
To see that G_k has no triangles, observe that any two adjacent edges incident to U have endpoints in distinct copies of G_{k-1}, thus are not part of any triangle. □

Theorem 54 (Kőnig, 1916)**.** If G is a bipartite graph with maximum degree Δ then $\chi'(G) = \Delta$.

Proof. We see, that $\chi'(G) \geq \Delta$ because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring. To prove that $\chi'(G) \leq \Delta$, we use induction on $||G||$ with a basis $||G|| = 1$. Let G be given, $||G|| \geq 2$, and assume that the statement is true for any graph on at most $||G|| - 1$ edges. Let $e = xy \in E(G)$. By induction, there is a proper edge coloring c of $G' = G - e$ using colors from $\{1, \ldots, \Delta\}$.

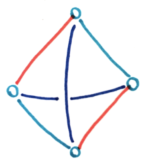
In G' both x and y are incident to at most $\Delta - 1$ edges. Thus, there are non-empty color sets $\text{Mis}(x), \text{Mis}(y) \subseteq [\Delta]$, where $\text{Mis}(v)$ is the set of "missing" colors, i.e., the set of colors that are not used on edges incident to v and $v \in \{x, y\}$.
If $\text{Mis}(x) \cap \text{Mis}(y) \neq \emptyset$, let $\alpha \in \text{Mis}(x) \cap \text{Mis}(y)$, color e with α. This gives $\chi'(G) \leq \Delta$.
If $\text{Mis}(x) \cap \text{Mis}(y) = \emptyset$, let $\alpha \in \text{Mis}(x)$ and $\beta \in \text{Mis}(y)$, consider the longest path P colored α and β starting at x. Because of parity, P does not end in y, and because y is not incident to β, y is not a vertex on P. Switch colors α and β on P. Then we obtain a proper edge-coloring in which $\beta \in \text{Mis}(x) \cap \text{Mis}(y)$, which allows e to be colored β. Thus $\chi'(G) \leq \Delta$.

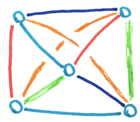
□



Theorem 5.4 (Vizing's Theorem, 5.3.2)**.**
For any graph G with maximum degree Δ,

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$





Proof. The lower bound holds because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring. For the upper bound we use induction on $||G||$ with the trivial basis $||G|| = 1$. Let G be a graph, $||G|| > 1$, assume that the assertion holds for all graphs with smaller number of edges. For any edge-coloring c of a subgraph H of G with colors $[\Delta + 1]$, and for any vertex v, let $\text{Mis}_c(v)$ denote the set of colors from $[\Delta + 1]$ not used on the edges of H incident to v. Assume now that G has no proper edge-coloring with $\Delta + 1$ colors.

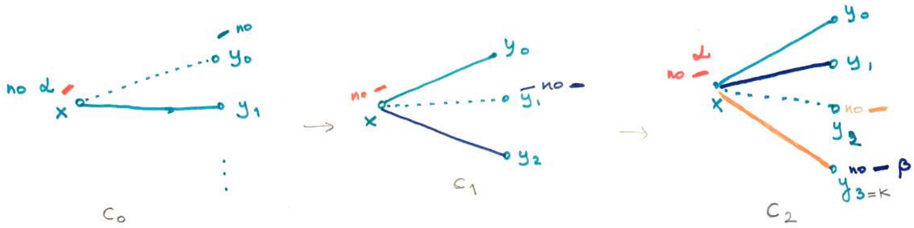
Claim. For any $e = xy \in E(G)$, for any proper coloring c of $G - e$ from $[\Delta + 1]$, for any $\alpha \in \text{Mis}_c(x)$ and any $\beta \in \text{Mis}_c(y)$, there is an x-y-path colored α and β.

We see that $\text{Mis}_c(v) \neq \emptyset$ for any v. If $\text{Mis}_c(x) \cap \text{Mis}_c(y) \neq \emptyset$, let $\alpha \in \text{Mis}_c(x) \cap \text{Mis}_c(y)$. Color xy with α, this gives a proper coloring of G with at most $\Delta + 1$ colors, a contradiction.

If $\text{Mis}_c(x) \cap \text{Mis}_c(y) = \emptyset$, let $\alpha \in \text{Mis}_c(x)$, $\beta \in \text{Mis}_c(y)$, $\alpha \neq \beta$. If there is maximal path P colored α and β that contains x and does not contain y, switch the colors α and β in P and color xy with β. This gives a proper coloring of G with at most $\Delta + 1$ colors, a contradiction. This proves the Claim.

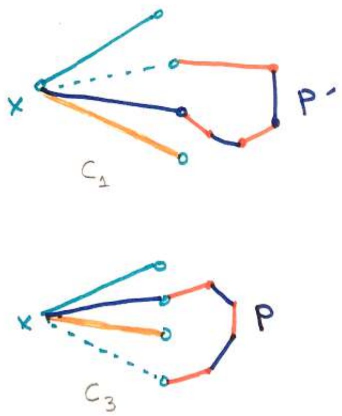
Let $xy_0 \in E(G)$. Let c_0 be a proper coloring of $G_0 := G - xy_0$ from $[\Delta + 1]$. Let $\alpha \in \text{Mis}_{c_0}(x)$. Let $y_0, y_1, \ldots, y_k$ be a maximal sequence of distinct neighbors of x such that $c_0(xy_{i+1}) \in \text{Mis}_{c_0}(y_i)$, $0 \leq i < k$.

Let c_i be a coloring of $G_i := G - xy_i$ such that $c_i(xy_j) = c_0(xy_{j+1})$, for $j \in \{0, \ldots, i-1\}$; $c_i(e) = c_0(e)$, otherwise. Note that $\text{Mis}_{c_i}(x) = \text{Mis}_{c_j}(x)$, for all $i, j \in \{0, \ldots, k\}$.



Let $\beta \in \mathrm{Mis}_{c_0}(y_k)$. Let $y = y_i$ be a vertex so that $c_0(yx) = \beta$. Such a vertex exists, otherwise either $\beta \in \mathrm{Mis}_{c_k}(y_k) \cap \mathrm{Mis}_{c_k}(x)$, contradicting Claim, or the sequence $y_0, \dots, y_k$ can be extended, contradicting its maximality.

Then G_k has an α-β path P with endpoints y_{i-1}, y_k in $G_k - x$. On the other hand G_i has an α-β path P' with endpoints y_{i-1}, y_i in $G_i - x$. Since $G - x$ is colored identically in c_k and c_i, we have that $P \cup P'$ is a two-colored graph, connected since both paths P and P' contain y_{i-1} and having three vertices of degree 1. This is impossible.



□

Lemma 55. The list chromatic number of $G = K_{n,n}$ with $n = \binom{2k}{k}$ is at least $k+1$.

Proof. Let L be a list assignment to the vertices of G with each list of size k such that the set of lists for parts A and B in G is $\binom{[2k]}{k}$. We shall show that G is not colorable from these lists. Assume the opposite, i.e. that c is a proper L-coloring. Let $v_1 \in A$ have color a_1. Let $v_2 \in A$ have a list $L(v_2)$ not containing a_1, let $c(v_2) = a_2$, $a_2 \neq a_1$. Assume $v_1, \dots, v_i$ are vertices of A of distinct colors $a_1, \dots, a_i$, respectively, $i < k$. Let v_{i+1} be a vertex in A such that $L(v_{i+1}) \cap \{a_1, \dots, a_i\} = \emptyset$. Such a vertex exists because $|[2k] - \{a_1, \dots, a_i\}| \geq k$, so v_{i+1} can be taken to be a vertex with a list that is a subset of $[2k] - \{a_1, \dots, a_i\}$. Consider $v_1, \dots, v_k$ of distinct colors $a_1, \dots, a_k$.

Consider a vertex $u \in B$ such that $L(u) = \{a_1, \ldots, a_k\}$. Then u can not be colored from its list. $\square$

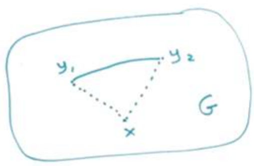
Theorem 56 (Galvin's Theorem). Let G be a bipartite graph. Then $\mathrm{ch}'(G) = \chi'(G)$.

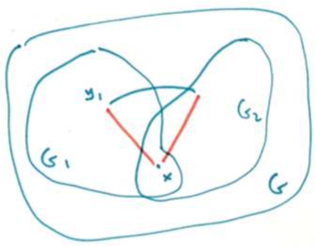
Definition. A graph is *k-constructible* if it is isomorphic to K_k or it is obtained from vertex-disjoint k-constructible graphs G_1, G_2 via one of the following operations: a) contraction of two non-adjacent vertices of G_1, b) identifying one vertex from G_1 with a vertex in G_2, call it x, deleting an edge xy_i in G_i, $i = 1, 2$, adding the edge $y_1 y_2$.

Theorem 57 (Hajós 1961). A graph has chromatic number at least k if and only if it contains a k-contructible subgraph.

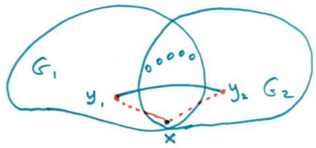
Proof. Assume first that our graph contains a k constructible subgraph G. If $G = K_k$ then $\chi(G) = k$. If G is constructed from a k-constructible graph G_1 by contracting two non-adjacent vertices x and y into v_{xy}, such that $\chi(G_1) \geq k$, then $\chi(G) \geq k$. Indeed, otherwise a proper ℓ-coloring of G with $\ell < k$ can be used to create a proper ℓ-coloring of G_1 by using the color of v_{xy} on both x and y. Let G be created from k-constructible graphs G_1, G_2, $\chi(G) \geq k$, $i = 1, 2$, by identifying one vertex from G_1 with a vertex in G_2 (call it x), deleting an edge xy_i in G_i, $i = 1, 2$, and adding an edge $y_1 y_2$. Then if $\chi(G) < k$, consider a proper coloring c of G with less than k colors. Under this coloring y_1 and y_2 get distinct colors, so one of them get a color different from $c(x)$, say $c(y_1) \neq c(x)$. Thus c restricted to $V(G_1)$ is a proper coloring of G_1 with less than k colors, a contradiction. So, we have that $\chi(G) \geq k$.

Now assume that $\chi(G) = k$. Assume that G does not contain k-constructible subgraphs, add edges greedily as long as this property holds. Let the resulting graph be G'. We see that adding any edge to G' creates a k-constructible subgraph. If G' is a clique, then it has k vertices and it is constructible, a contradiction. If G' is complete multipartite, it has k parts, so it contains K_k, that is constructible, a contradiction. If G' is not complete multipartite, there are vertices x, y_1, y_2, such that $y_1 y_2 \in E(G')$ and $xy_1, xy_2 \notin E(G')$.

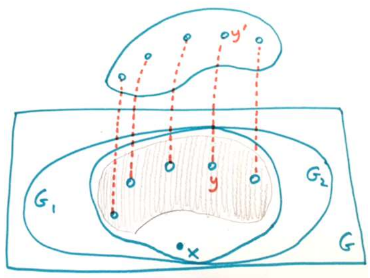




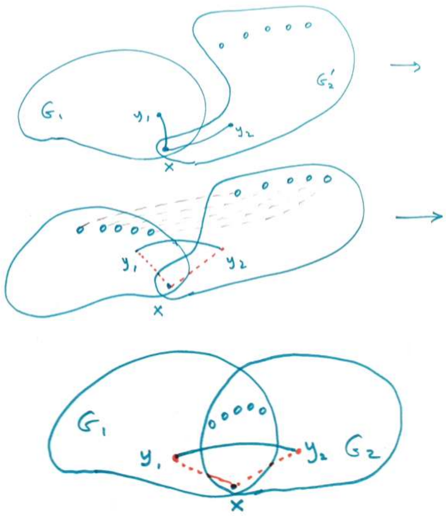
We see that $G + xy_i$ contains a k-constructible graph G_i, $i = 1, 2$.



Let G_2' be a copy of G_2 on a vertex set $\{y' : y \in V(G_2)\}$, such $y' = y$ iff $y = x$ and $y \in G_2 - G_1$, otherwise $y' \notin V(G_1)$. Assume that y' plays a role of y.



Then G_1 and G_2' are k-constructible and a graph $G^* = G_1 \cup G_2 - \{xy_1, xy_2'\} \cup \{y_1 y_2'\}$ is k-constructible.



Contracting y with y' in G^* for all $y \neq y'$, $y \in V(G_2)$ results in a k-contractible graph. This graph is a subgraph of G', a contradiction. $\square$

Other coloring results

Total colorings
Let $\chi''(G)$ be the smallest number of colors one can assign to vertices and edges of G such that no two adjacent and no two incident elements have the same color. This parameter is called *total chromatic number.*

Vizing conjectured that $\chi''(G) \le \Delta(G) + 2$. One of the best known upper bounds is due to Molloy and Reed: $\chi''(G) \le \Delta(G) + 10^{26}$.

Edge colorings of multigraphs
Let G be a multigraph with no loops and each edge repeated at most μ times. Then $\chi'(G) \le (3/2)\Delta(G)$ and $\chi'(G) \le \Delta(G)+\mu$, where $\Delta(G)$ is the largest number of edges, counting multiplicities, incident to a vertex of G.

Chromatic number, max degree Δ, and clique number ω
Reed's Conjecture:

$$\chi(G) \le \left\lceil \frac{1 + \Delta(G) + \omega(G)}{2} \right\rceil .$$

It is known to be true for $\omega \in \{2, \Delta - 1, \Delta, \Delta + 1\}$.

Johanssen proved that there is a constant $C > 0$ such that if G is triangle-free and $\Delta = \Delta(G)$, then $\chi(G) \le C \frac{\Delta}{\log \Delta}$.

List-colorings and list-edge-colorings
We have seen that χ and $\chi_\ell = ch$ can be very far apart for some graphs, for example for large complete bipartite graphs. However, it is conjectured that the situation is very different for list-edge-chromatic number, namely that $\chi'_\ell(G) = \chi'(G)$ for any graph G. It is proved to be true for bipartite graphs G by Galvin.

When a graph has a large chromatic number compared to the number of vertices, then a similar result holds for vertex colorings. It was proved by Noel, Reed, and Wu in 2005 that if $|V(G)| \le 2\chi(G)+1$ then $\chi_\ell(G) = \chi(G)$. This proved a famous conjecture by Ohba. However, for dense graphs, the list-chromatic number is always large, even if the graph itself has a small chromatic number. Indeed, Alon proved in 1993 that for each natural number k there is a natural number $f(k)$ such that for any G with average degree at least $f(k)$, $\chi_\ell(G) \ge k$.

Chromatic number of hypergraphs
A vertex coloring of a hypergraph is proper if there is no monochromatic edge. The smallest number of colors in a proper vertex coloring of a hypergraph is called its chromatic number. A hypergraph has *property B* due to Felix Bernstein (1908), if its chromatic number is 2.

A hypergraph is r-uniform if each hyperedge has size r. A Berge-cycle of length k is a hypergraph with k distinct edges $e_0, \ldots, e_{k-1}$ containing vertices $v_0, \ldots, v_{k-1}$ such that $v_i, v_{i+1} \in e_i$, $i = 0, \dots k-1$, addition mod k. The girth of a hypergraph is the length of a shortest cycle contained in the hypergraph as a subgraph.

Lovász proved in 1968 that for any $r, k, \ell \geq 2$ there is a hypergraph with chromatic number k, girth ℓ and uniformity r.

Erdős introduced the function $m(n)$ that is the smallest number of edges in an n-uniform hypergraph that does not have property B. The following bounds are known,

$$C 2^n \sqrt{\frac{n}{\log n}} \leq m(n) \leq C' 2^n n^2.$$

The upper bound was proved by Erdős in 1964 and the lower bound is due to Radhakrishnan and Srinivasan, 2000.

Conjecture (Hadwiger Conjecture)**.** Let r be a natural number and G be a graph. Then $\chi(G) \geq r$ implies $MK_r \subseteq G$.

For $r \in \{1, 2, 3\}$ this is easy to see, and it is not too difficult to prove it for $r = 4$. For $r \in \{5, 6\}$ the conjecture has been proven using the 4-color-theorem. It is still open for $r \geq 7$.

In 2019, Norin and Song proved that any graph with no K_r minor is $O(r(\log r)^{0.354})$-colorable. The ideas of this proof were shortly after extended by Postle, who showed that any graph with no K_r minor is $O(r(\log r)^{\beta})$-colorable for any $\beta > 1/4$. These are currently the best results (in general) towards Hadwiger's conjecture.

6 Extremal graph theory

In this section $c, c_1, c_2, \ldots$ always denote unspecified constants in $\mathbb{R}_{>0}$.

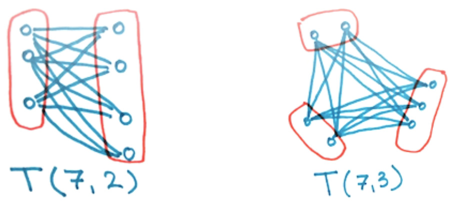
Definition 6.1.

- Let n be a positive integer and H a graph. The *extremal number* $\mathrm{ex}(n, H)$ denotes the maximum size of a graph of order n that does not contain H as a subgraph and $\mathrm{EX}(n, H)$ is the set of H-free graphs on n vertices with $\mathrm{ex}(n, H)$ edges.

  extremal number, $\mathrm{ex}(n, H)$
  $\mathrm{EX}(n, H)$

- Let n and r be integers with $1 \leq r \leq n$. The *Turán graph* $T_r(n)$ is the unique complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}. We denote $\|T_r(n)\|$ by $t_r(n)$.

  Turán graph, $T_r(n)$
  $t_r(n)$



- In the special case that $n = r \cdot s$, for positive integers n, r, s with $1 \leq r \leq n$, the Turán graph $T_r(n)$ is also denoted by K_r^s.

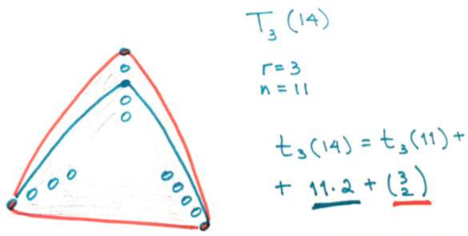
  K_r^s

Example.

- $\mathrm{ex}(n, K_2) = 0$, $\mathrm{EX}(n, K_2) = \{E_n\}$
- $\mathrm{ex}(n, P_3) = \lfloor n/2 \rfloor$, $\mathrm{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$

Lemma 58. For any $r, n \geq 1$, $t_r(n + r) = t_r(n) + n(r - 1) + \binom{r}{2}$.

Proof. Consider $G = T_r(n + r)$ graph with parts $V_1, \ldots, V_r$. Let $v_i \in V_i$, $i = 1, \ldots, r$. Then $G' = G - \{v_1, \ldots, v_r\}$ is isomorphic to $T_r(n)$. We have that $||G|| - ||G'||$ is equal to the number of edges incident to v_i's, for some $i = 1, \ldots, r$. This number is $n(r - 1) + \binom{r}{2}$.



□

Lemma 59. Among all n-vertex r-partite graphs, $T_r(n)$ has the largest number of edges.

Proof. Let first $r = 2$. Let G be an n-vertex bipartite graph with largest possible number of edges. Then clearly G is complete bipartite. Assume that two parts V and U of G differ in size by at least 2, so $|V| > |U| + 1$. Put one vertex from V to U to obtain new parts V' and U' and let G' be complete bipartite graph with parts V' and U'. Then $||G'|| = |V'||U'| = (|V| - 1)(|U| + 1) = |V||U| - |U| + |V| - 1 > |V||U| - |U| + |U| + 1 - 1 = |V||U| = ||G||$, a contradiction to maximality of G.

Now, if $r > 2$, consider any two parts U, V of an r-partite G. Assume that U differs from V by at least 2 in size. Let X be the remaining set of vertices. Then $||G|| = ||G[X]|| + |X|(n - |X|) + ||G[U \cup V]||$. Let G' be a graph on the same set of vertices as G that differs from G only on edges induced by $U \cup V$ and so that $G'[U \cup V]$ is a balanced complete bipartite graph. Then from the previous paragraph with $r = 2$, we see that $||G'[U \cup V]|| > ||G[U \cup V]||$. Thus $||G'|| > ||G||$, a contradiction. Thus any two parts of G differ in size by at most 1. In addition we see as before that G is complete r-partite. Thus G is isomorphic to $T_r(n)$. □

Lemma 60. For a fixed r,

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}.$$

Proof. Since each part in $T_r(n)$ has size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$, we see that each part has size between $(n - r)/r$ and $(n + r)/r$. We have that

$$\binom{n}{2} - r\binom{(n+r)/r}{2} \le t_r(n) \le \binom{n}{2} - r\binom{(n-r)/r}{2}.$$

Thus

$$\binom{n}{2} - r\frac{1}{2}\frac{(n+r)}{r}\frac{n}{r} \le t_r(n) \le \binom{n}{2} - r\frac{1}{2}\frac{(n-r)}{r}\frac{(n-2r)}{r}.$$

This gives

$$\binom{n}{2} - \frac{1}{2}(n+r)\frac{n}{r} \le t_r(n) \le \binom{n}{2} - \frac{1}{2}(n-r)\frac{(n-2r)}{r}.$$

Dividing each term by $\binom{n}{2}$ gives.

$$1 - \frac{1}{r}\frac{(n+r)n}{n(n-1)} \le \frac{t_r(n)}{\binom{n}{2}} \le 1 - \frac{1}{r}\frac{(n-r)(n-2r)}{n(n-1)}.$$

□

Let H be a t-uniform hypergraph and $X \subseteq V(H)$ with $|X| \ge t$. Then X induces a clique in H if it induces $\binom{|X|}{t}$ edges.

Lemma 61. Let $\mathcal{H}$ be a set of t-uniform hypergraphs, $t \geq 2$, on n vertices that is a pairwise vertex-disjoint union of k cliques. Then a hypergraph in $\mathcal{H}$ with the smallest number of hyperedges is the one where all cliques have almost equal sizes.

Proof. Assume that $H \in \mathcal{H}$ has two cliques on vertex sets of sizes a, b, $b \geq a+2$, i.e. $a < b-1$. Move one vertex from the larger to the smaller clique and consider two cliques on vertex sets of sizes $a' = a+1$, $b' = b-1$. Consider the difference d between the number of hyperedges in the two new and the two old cliques

$$\begin{aligned}
d &= \binom{a'}{t} + \binom{b'}{t} - \binom{a}{t} - \binom{b}{t} = \binom{a+1}{t} + \binom{b-1}{t} - \binom{a}{t} - \binom{b}{t} \\
&= \frac{1}{t!}\left((a+1)(a)\cdots(a-t+2) + (b-1)(b-2)\cdots(b-t)\right) - \\
&\quad \frac{1}{t!}\left(a(a-1)\cdots(a-t+1) - b(b-1)\cdots(b-t+1)\right) \\
&= \frac{1}{t!}\left(a(a-1)\ldots(a-t+2)[a+1-(a-t+1)]\right) + \\
&\quad \frac{1}{t!}\left((b-1)\ldots(b-t+1)[b-t-b]\right) \\
&= \frac{1}{t!}\left(a(a-1)\ldots(a-t+2)t - (b-1)\ldots(b-t+1)t\right) \\
&= \frac{t}{t!}\left(a(a-1)\ldots(a-t+2) - (b-1)\ldots(b-t+1)\right) \\
&< 0.
\end{aligned}$$

This contradicts the fact that H had the smallest number of hyperedges. $\square$

Theorem 62 (Mantel's theorem)**.** If a graph G on n vertices contains no triangle then it contains at most $\frac{n^2}{4}$ edges.

First proof of Mantel's theorem. We proceed by induction on n. For $n = 1$ and $n = 2$, the result is trivial, so assume that $n > 2$ and we know it to be true for $n-1$. Let G be a graph on n vertices. Let x and y be two adjacent vertices in G. Since every vertex in G is connected to at most one of x and y, there are at most $n-2$ edges between $\{x, y\}$ and $V(G) - \{x, y\}$. Let $H = G - \{x, y\}$. Then H contains no triangles and thus, by induction, H has at most $(n-2)^2/4$ edges. Therefore, the total number of edges in G is at most $(n-2)^2/4 + n - 1 = n^2/4$. $\square$

Second proof of Mantel's theorem. Let A be the largest independent set in the graph G. Since the neighborhood of every vertex x is an independent set, we must have $\deg(x) \leq |A|$. Let B be the complement of A. Every edge in G must meet a vertex of B. Therefore, the number of edges in G satisfies $||G|| \leq \sum_{x \in B} \deg(x) \leq |A||B| \leq (|A| + |B|)^2/4 = n^2/4$. $\square$

Note that the equality holds for even n if and only if $|A| = |B|$ and G is a complete bipartite graph with parts A and B.

Theorem 6.2 (Turán's Theorem, 7.1.1)**.** For all integers $r > 1$ and $n \geq 1$, any graph G with n vertices, $\mathrm{ex}(n, K_r)$ edges and $K_r \not\subseteq G$ is a $T_{r-1}(n)$. In other words $\mathrm{EX}(n, K_r) = \{T_{r-1}(n)\}$.

Proof. We shall use induction on n for a fixed r. If $n \leq r-1$, then K_n is the graph with largest number of edges, n vertices and no copy of K_r. Since $K_n = T_{r-1}(n)$, the basis case is complete.

Assume that $n > r-1$. Let $G \in EX(n, K_r)$. Then G contains a copy K of K_{r-1} otherwise we could add an edge to G without creating a copy of K_r, thus violating maximality of G. Let $G' = G - V(K)$. By induction hypothesis

$$||G'|| \leq t_{r-1}(n-r+1).$$

Thus

$$||G|| = ||G'||+||K||+||G[V(K), V-V(K)]|| \leq t_{r-1}(n-r+1)+\binom{r-1}{2}+(n-r+1)(r-2). \tag{1}$$

Indeed, the last term holds since any vertex of $V - V(K)$ is adjacent to at most $|V(K)| - 1 = r-2$ vertices of K (otherwise we would have had a copy of K_r in G).

By Lemma 58, $t_{r-1}(n) = t_{r-1}(n-r+1) + \binom{r-1}{2} + (n-r+1)(r-2)$ and thus

$$||G|| \leq t_{r-1}(n).$$

On the other hand we know that $T_{r-1}(n)$ does not have K_r as a subgraph, so the densest K_r-free graph G should have at least as many edges as $T_{r-1}(n)$.

Thus

$$||G|| \geq t_{r-1}(n).$$

In particular

$$||G|| = t_{r-1}(n)$$

and all inequalities in (1) are equalities.

So, in particular $||G'|| = t_{r-1}(n-r+1)$, by induction $G' = T_{r-1}(n-r+1)$, and each vertex of $V - V(K)$ sends exactly $r-2$ edges to K. Let $V_1, \ldots, V_{r-1}$ be the parts of G'. For all $v \in V_1 \cup \cdots \cup V_{r-1}$, let $f(v) \in V(K)$ so that v is not adjacent to $f(v)$.

If there are indices $i, j \in [r-1]$, $i \neq j$ so that there are vertices $v \in V_i$, $v' \in V_j$ for which $f(v) = f(v')$, then $V(K) \cup \{v, v'\}$ induces an r-clique in G, a contradiction.

Therefore we can suppose that for all $i, j \in [r-1]$, $i \neq j$ and for any $v \in V_i$ and $v' \in V_j$, $f(v) \neq f(v')$. It implies that for any $i \in [r-1]$ and any $u, u' \in V_i$, $f(u) = f(u')$. Denote the vertices of K by $v_1, \ldots, v_{r-1}$ where v_i $f(u_i)$ with $u_i \in V_i$.

Then $G = T_{r-1}(n)$ with parts $V_i \cup \{v_i\}$.

□

Theorem 63. For any positive integers n and k, $n \geq k$,

$$\operatorname{ex}(n, P_{k+1}) \leq \frac{k-1}{2} n.$$

Moreover, if n is divisible by k then the equality holds. In addition, for any $n \geq k$ there is an extremal graph $G \in EX(n, P_{k+1})$ such that G is pairwise vertex disjoint union of cliques all of which have size k except for at most one of size at most k.

Proof. We prove only the first two statements.
We shall prove that $\operatorname{ex}(n, P_{k+1}) \leq \frac{k-1}{2} n$ by induction on n.
If $n = k$ then K_n contains no copy of P_{k+1}, $||K_n|| = \binom{n}{2} = n(n-1)/2 = n(k-1)/2$.
Assume that $n > k$. Let G be a graph on n vertices not containing a path of length k as a subgraph. If G is not connected, i.e., G is a union of two vertex disjoint graphs G_1 and G_2 on t and $n-t$ vertices, respectively, $0 < t < n$, then by induction

$$||G|| = ||G_1|| + ||G_2|| \leq \frac{k-1}{2} t + \frac{k-1}{2}(n-t) = \frac{k-1}{2} n.$$

So, we assume that G is connected. We shall prove first that $\delta(G) \leq (k-1)/2$. Assume not and consider a longest path P in G with end-points x and y. Then $N(x), N(y) \subseteq V(P)$. Since $|N(x)|, |N(y)| > (k-1)/2$ and $||P|| < k$, there are consecutive vertices x', y' on P such that $xy', x'y \in E(G)$, and so $xPx'yPy'x$ is a cycle C. If there is an edge in G with one vertex on C and another not, we can find a longer path, a contradiction. Since G is connected, we have then that $V(C) = V(G)$, a contradiction since $|G| = n > k \geq |V(C)|$.

Let x be a vertex of minimum degree in G. Thus

$$||G|| \leq (k-1)/2 + ||G - x|| \leq \frac{k-1}{2} + \frac{k-1}{2}(n-1) = \frac{k-1}{2} n.$$

On the other hand, when n is divisible by k, observe that a pairwise vertex-disjoint union of cliques on k vertices does not contain a path of length k as a subgraph and has a desired number $(n/k)\binom{k}{2} = (k-1)n/2$ edges.

□

Theorem 64. Let G be a graph on n vertices and at least kn edges, $k < n/2$. Then G contains all k-vertex trees as subgraphs.

Proof. First we note that there is a subgraph G' of G of minimum degree at least k. Indeed, otherwise there is a vertex v_1 of degree at most $k-1$, in $G - v_1$ there is a vertex v_2 of degree at most $k-1$, etc. So, the total number of edges then is at most $n(k-1)$, a contradiction.

We shall show that G' contains all k-vertex trees as subgraphs by induction on k. If $k = 1$, then the statement is trivial. Assume that $k > 1$. Let T be a tree on k vertices and let $T' = T - v$, where v is a leaf of T. Let u be the neighbor of v in T. Then by induction G' contains a copy of T' with a vertex u' playing a role of u. Since $\deg_{G'}(u') \geq k$ and $|T' - u'| = k - 2$, we see that there is a vertex of $v' \in V(G' - T')$, such that v' is adjacent to u'. Thus $V(T') \cup \{v'\}$ induces a graph containing a copy of T. □

Conjecture 65 (Erdős-Sós)**.** If $|G| = n$ and $||G|| > (k-1)n/2$, then G contains all k-edge trees as subgraphs. I.e., for any tree T on k edges $\mathrm{ex}(n, T) \leq \frac{(k-1)n}{2}$.

Theorem 66 (Erdős-Stone-Simonovits)**.** For any graph H and for any fixed $\epsilon > 0$, there is n_0 such that for any $n \geq n_0$,

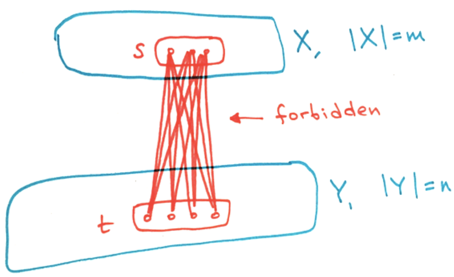
$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right)\binom{n}{2} \leq \mathrm{ex}(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right)\binom{n}{2}.$$

Proof outline. Let $r = \chi(H) - 1$.

For the upper bound, let G be a graph on n vertices that has $\left(1 - \frac{1}{\chi(H)-1} + \epsilon\right)\binom{n}{2}$ edges. We shall show that G has a subgraph isomorphic to H. Let G' be a large subgraph of G that has minimum degree at least $(1 - 1/r + \epsilon/2)|V(G')|$, we can find such a G' by greedily deleting vertices of smaller degrees. Then show, by induction on r that G' contains a complete $(r+1)$-partite graph H' with sufficiently large parts. Finally, observe that $H \subseteq H'$.

For the lower bound, observe that $T_r(n)$ does not contain H as a subgraph and has the desired number of edges. □

Definition 6.3. The *Zarankiewicz* function $z(m, n; s, t)$ denotes the maximum number of edges that a bipartite graph with parts X, Y of sizes m, n, respectively, can have without containing $K_{s,t}$ respecting sides (i.e., there is no copy of $K_{s,t}$ with partition sets S, T, of sizes s, t, respectively, such that $S \subseteq X$ and $T \subseteq Y$). Zarankiewicz, $z(m, n; s, t)$



Theorem 67 (Kővári-Sós-Turán Theorem)**.**
We have the upper bound

$$z(m, n; s, t) \leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$$

for the Zarankiewicz function. In particular,

$$z(n, n; t, t) \leq c_1 \cdot n \cdot n^{1-1/t} + c_2 \cdot n = \mathcal{O}(n^{2-1/t})$$

for $m = n$ and $t = s$.

Proof. Let G be a bipartite graph with parts A, $|A| = m$ and B, $|B| = n$ such that it does not contain a copy of $K_{s,t}$ with part of size s in A and part of size t in B. Let T be the number of stars of size t with a center in A. Then

$$T = \sum_{v \in A} \binom{\deg(v)}{t}.$$

On the other hand

$$T \leq (s-1)\binom{n}{t}.$$

Since for each subset Q of t vertices in B there are at most $s-1$ stars counted by T with a leaf-set Q. Thus

$$\sum_{v \in A} \binom{\deg(v)}{t} \leq (s-1)\binom{n}{t}.$$

Let $e = ||G||$. Then $e = \sum_{v \in A} \deg(v)$. Then by Lemma 61.

$$\sum_{v \in A} \binom{\deg(v)}{t} \geq m\binom{e/m}{t}.$$

Thus

$$\begin{aligned}
m\binom{e/m}{t} &\leq (s-1)\binom{n}{t} \Longrightarrow \\
\frac{m}{s-1} &\leq \frac{\binom{n}{t}}{\binom{e/m}{t}} \Longrightarrow \\
\frac{m}{s-1} &\leq \frac{n}{e/m}\frac{n-1}{e/m-1}\cdots\frac{n-t+1}{e/m-t+1} \Longrightarrow \\
\frac{m}{s-1} &\leq \left(\frac{n-t+1}{e/m-t+1}\right)^t \Longrightarrow \qquad (2) \\
e &\leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.
\end{aligned}$$

Note here, that (2) holds since $p/q < (p-1)/(q-1)$ iff $p > q$. Here $p = n$ and $q = e/m$ and we have that $e \leq mn$, so $e/m < n$.

□

Lemma 68. For any positive integers n, t, $t < n$, $\mathrm{ex}(n, K_{t,t}) \leq z(n, n; t, t)/2$.

Proof. Let G be a graph on n vertices with no subgraph isomorphic to $K_{t,t}$. Let G' be a bipartite graph with partite sets $V(1), V(2)$, $V(i) = \{v(i) : v \in V(G)\}$ and and edge set $E = \{v(1)u(2) : uv \in E(G)\}$. Then we see that $||G'|| = 2||G||$. Assume that there is a copy of $K_{t,t}$ in G' with parts $V'(1) \subseteq V(1)$, $V'(2) \subseteq V(2)$. Then if $v(1) \in V'(1)$, $u(2) \in V'(2)$, then $u \neq v$. Thus this copy of $K_{t,t}$ corresponds to a $K_{t,t}$ in G. Therefore $||G'|| \leq z(n, n; t, t)$ which completes the proof. $\square$

Theorem 69. For any positive t, and $n > t$, there are positive constants C and C' such that
$$C' n^{2-\frac{2}{t+1}} \leq \mathrm{ex}(n, K_{t,t}) \leq C n^{2-1/t}.$$

Proof. The upper bound follows from Theorem 67 and Lemma 68.

Thus we shall only prove the lower bound. Let $G = G(n, p)$ be a random graph where the edges are chosen independently with probability p each. Let $p = n^{-2/(t+1)}$. Then $Exp(|E(G)|) = p\binom{n}{2}$ and $Exp(\#K_{t,t}\ 's) \leq \binom{n}{2t}\binom{2t}{t}p^{t^2}$. Delete an edge from each copy of $K_{t,t}$. Call the resulting graph G'. Note that G' has no copies of $K_{t,t}$.

$$\begin{aligned}
Exp(|E(G')|) &\geq Exp(|E(G)|) - Exp(\#K_{t,t}\ 's) \\
&\geq p\binom{n}{2} - \binom{n}{2t}\binom{2t}{t}p^{t^2} \\
&\geq n^{2-2/t+1} - n^{2t}(t!)^{-2}n^{-2t^2/(t+1)} \\
&= n^{2-2/(t+1)} - n^{2t-2t+2t/(t+1)}(t!)^{-2} \\
&= n^{2-2/(t+1)}(1 - 1/2) \\
&= C' n^{2-2/(t+1)}.
\end{aligned}$$

Thus there is a graph with at least $C' n^{2-2/(t+1)}$ edges and no copy of $K_{t,t}$. $\square$

Corollary 70. If $\chi(H) \geq 3$, then $\mathrm{ex}(n, H) = cn^2(1 + o(1))$, for some constant c. If $\chi(H) = 2$, then $\mathrm{ex}(n, H) = o(n^2)$.

Theorem 71 (Erdős, Rényi, Sós; Bondy and Simonovits; Lazebnik, Ustimenko, Woldar)**.**

$$\mathrm{ex}(n, C_4) = \frac{1}{2}n^{3/2} + o(n^{3/2}),\ \mathrm{ex}(n, C_6) = \Theta(n^{4/3}), \mathrm{ex}(n, C_{10}) = \Theta(n^{6/5}),$$

$$C' n^{1+\frac{2}{3k-2-\epsilon}} \leq \mathrm{ex}(n, C_{2k}) \leq C n^{1+\frac{1}{k}},$$

where $\epsilon = 0$ if k is even, $\epsilon = 1$ if k is odd.

Proof. We shall only prove that $C' n^{3/2} \leq \mathrm{ex}(n, C_4) \leq C n^{3/2}$ for some positive constants C and C'. The upper bound is implied by Theorem 69.

For the lower bound, we need to find a graph on n vertices, $C'n^{3/2}$ edges, and not containing C_4 as a subgraph.

First we shall contruct a C_4-free graph H_p on $p(p-1)$ vertices for a prime p. Let $V(H_p) = \mathbb{Z}_p \setminus \{0\} \times \mathbb{Z}_p$. Two vertices (a,b) and (c,d) are adjacent if and only if $ac = b+d$ modulo p.

Assume first that H_p contains a copy of C_4 - $(c,d),(x',y'),(a,b),(x'',y''),(c,d)$. Then the system

$$\begin{cases} ax = b+y \\ cx = d+y \end{cases}$$

has two distinct solutions $(x,y) = (x',y')$ and $(x,y) = (x'',y'')$. However, subtracting the equations of the system we get $(a-c)x = b-d$. If $b-d = 0$, then since $(a,b) \neq (c,d)$, $a-c \neq 0$, so $x = 0$, impossible. If $b-d \neq 0$, then since $x \neq 0$, $a-c \neq 0$ and then $x = (b-d)(a-c)^{-1}$. So, x is defined uniquely. Then $y = ax-b$ is also defined uniquely. A contradiction to our assumption that there are two solutions. Now, we shall find $||H_p||$. For each vertex (a,b) there are $p-1$ solutions of the equation $ax = b+y$. Indeed, choose x arbitrarily in $p-1$ ways and express y. Thus H_p is a $(p-1)$ regular graph on $p(p-1)$ vertices, so $||H_p|| = (p-1)^2p/2$. We see, that $||H_p|| \geq c|H_p|^{3/2}$.

Now, we need to construct a C_4-free graph G on n vertices for an arbitrary n, so that $||G|| \geq c'n^{3/2}$. We note that for any sufficiently large m there is a prime number p, $p \in (m - m^{0.6}, m]$. Let $m = \lfloor \sqrt{n} \rfloor$, pick a prime $p \in (m - m^{0.6}, m]$. Then

$$0.99n \leq (m - m^{0.6} - 1)^2 \leq p(p-1) \leq m^2 \leq n.$$

Let G be a graph consisting of H_p and isolated vertices. Clearly G does not have C_4's as subgraphs since H_p does not. In addition,

$$||G|| = ||H_p|| = c|H_p|^{3/2} \geq c(0.99n)^{3/2} = c'n^{3/2}.$$

$\square$

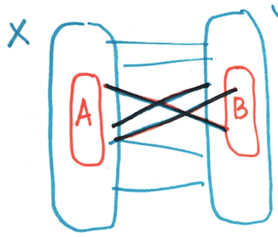
Theorem 72 (Sachs, Erdős; Imrich; Erdős-Gallai)**.** If $\delta(G) = d > 2$ then G contains a cycle of length at most $2\log n/\log(d-1)$. For any integer $d > 2$ there is a graph of minimum degree d that has no cycles of lengths at most $0.4801\log n/\log(d-1) - 2$. Any n vertex graph with $\frac{1}{2}(k-1)(n-1)$ edges has a cycle of length at least k. This is tight if $(n-1)$ is divisible by $(k-2)$.

Definition 6.4. Let $X, Y \subseteq V(G)$ be disjoint vertex sets and $\epsilon > 0$.

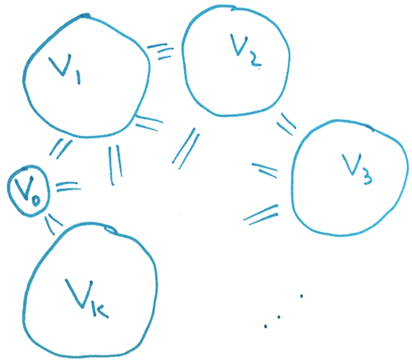
- We define $\|X, Y\|$ to be the number of edges between X and Y and *the density $d(X, Y)$ of $(X,\ Y)$* to be density, $d(X, Y)$

$$d(X, Y) := \frac{\|X, Y\|}{|X||Y|}.$$

- For $\epsilon > 0$ the pair (X, Y) is an *ϵ-regular pair* if we have $|d(X, Y) - d(A, B)| \leq \epsilon$ for all $A \subseteq X$, $B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$. ϵ-regular pair



- An *ϵ-regular partition* of the graph $G = (V, E)$ is a partition of the vertex set $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ with the following properties: ϵ-regular partition
  1. $|V_0| \leq \epsilon|V|$
  2. $|V_1| = |V_2| = \cdots = |V_k|$
  3. All but at most ϵk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ϵ-regular.



Theorem 6.5 (Szemerédi's Regularity Lemma, 7.4.1)**.** For any $\epsilon > 0$ and any integer $m \geq 1$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ϵ-regular partition $V_0 \dot\cup \cdots \dot\cup V_k$ with $m \leq k \leq M$.

Theorem 6.6 (Erdős-Stone Theorem, 7.1.2)**.** For all integers $r > s \geq 1$ and any $\epsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains K_r^s as a subgraph.

Corollary 73. Erdős-Stone together with $\lim_{n\to\infty} t(n,r)/\binom{n}{2} = 1 - 1/r$ yields an asymptotic formula for the extremal number of any graph H on at least one edge:

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

For example, $\mathrm{ex}(n, K_5 \setminus \{e\}) \simeq 2/3 \cdot \binom{n}{2}$ since $\chi(K_5 \setminus \{e\}) = 4$.

Chvátal and Szemerédi proved a more quantitative version of the Erdős-Stone theorem.

Theorem 74 (Chvátal-Szemerédi Theorem)**.** For any $\epsilon > 0$ and any integer $r \geq 3$, any graph on n vertices and at least $\big(1 - 1/(r-1) + \epsilon\big)\binom{n}{2}$ edges contains K_r^t as a subgraph. Here t is given by

$$t = \frac{\log n}{500 \cdot \log(1/\epsilon)}.$$

Furthermore, there is a graph G on n vertices and $\big(1 - (1+\epsilon)/(r-1)\big)\binom{n}{2}$ edges that does not contain K_r^t for

$$t = \frac{5 \cdot \log n}{\log(1/\epsilon)},$$

i.e., the choice of t is asymptotically tight.

Theorem 75 (Bollobás-Thomason 1998, 7.2.1)**.** Every graph G of average degree at least cr^2 contains K_r as a topological minor.

Theorem 76 (7.2.4)**.** Let G be a graph of minimum degree $\delta(G) \geq d$ and girth $g(G) \geq 8k+3$ for $d, k \in \mathbb{N}$ and $d \leq 3$. Then G has a minor H of minimum degree $\delta(H) \geq d(d-1)^k$.

Theorem 77 (Thomassen's Theorem, 7.2.5)**.** For all $r \in \mathbb{N}$ there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph of minimum degree at least 3 and girth at least $f(r)$ has a K_r minor.

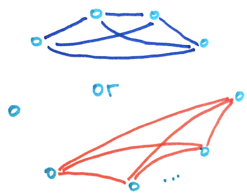
Theorem 78 (Kühn-Osthus, 7.2.6)**.** Let $r \in \mathbb{N}$. Then there is a constant $g \in \mathbb{N}$ such that we have $TK_r \subseteq G$ for every graph G with $\delta(G) \geq r-1$ and $g(G) \geq g$.

7 Ramsey theory

In every 2-coloring in this section we use the colors red and blue.

Definition 7.1.

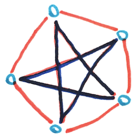
- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color, monochromatic
  - *rainbow* if no two edges have the same color, rainbow
  - *lexical* if two edges have the same color if and only if they have the same lower endpoint in some ordering of the vertices. lexical
- Let k be a natural number. Then the *Ramsey number* $R(k) \in \mathbb{N}$ is the smallest n such that every 2-edge-coloring of K_n contains a monochromatic K_k. Ramsey, $R(k)$

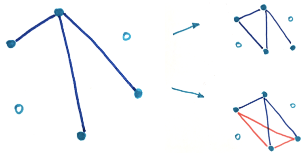


- Let k and l be natural numbers. Then the *asymmetric Ramsey number* $R(k, l)$ is the smallest $n \in \mathbb{N}$ such that every 2-edge-coloring of a K_n contains a red K_k or a blue K_l. asymmetric Ramsey, $R(k, l)$
- Let G and H be graphs. Then the *graph Ramsey number* $R(G, H)$ is the smallest $n \in \mathbb{N}$ such that every red-blue edge-coloring of K_n contains a red G or a blue H. graph Ramsey, $R(G, H)$
- Let $r, l_1, \ldots, l_k$ be natural numbers. Then the *hypergraph Ramsey number* $R_r(l_1, \ldots, l_k)$ is the smallest $n \in \mathbb{N}$ such that for every k-coloring of $\binom{[n]}{r}$ there is an $i \in \{1, \ldots, k\}$ and a $V \subseteq [n]$ with $|V| = l_i$ such that all sets in $\binom{V}{r}$ have color i. hypergraph Ramsey, $R_r(l_1, \ldots, l_k)$
- Let G and H be graphs. Then the *induced Ramsey number* $IR(G, H)$ is the smallest $n \in \mathbb{N}$ for which there is a graph F on n vertices such that in any red-blue coloring of $E(F)$, there is an induced subgraph of F isomorphic to G with all its edges colored red or there is an induced subgraph of F isomorphic to H with all its edges colored blue. induced Ramsey, $IR(G, H)$
- For $n \in \mathbb{N}$ and a graph H, the *anti-Ramsey number* $AR(n, H)$ is the maximum number of colors that an edge-coloring of K_n can have without containing a rainbow copy of H. anti-Ramsey, $AR(n, H)$

Lemma 79.

- $R(3) = 6$, i.e., every 2-edge-colored K_6 contains a monochromatic K_3 and there is a 2-coloring of a K_5 without monochromatic K_3's.

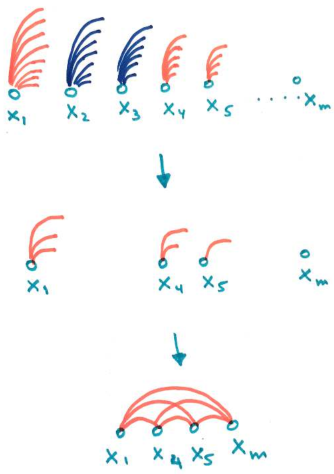




- Clearly, $R(2, k) = R(k, 2) = k$.

Theorem 7.2 (Ramsey Theorem, 9.1.1)**.** For any $k \in \mathbb{N}$ we have $\sqrt{2}^k \leq R(k) \leq 4^k$. In particular, the Ramsey numbers, the asymmetric Ramsey numbers and the graph Ramsey numbers are finite.

Proof. For the upper bound, consider an edge-coloring of $G = K_{4^k}$ with colors red and blue. Construct a sequence of vertices $x_1, \dots, x_{2k}$, a sequence vertex sets $X_1, \dots, X_{2k}$, and a sequence of colors $c_1, \dots, c_{2k-1}$ as follows. Let x_1 be an arbitrary vertex, $X_1 = V(G)$. Let X_2 be the largest monochromatic neighborhood of x_1 in X_1, i.e., largest subset of vertices from X_1, such that all edges from this subset to x_1 have the same color. Call this color c_1. We see that $|X_2| \geq \lceil \frac{|X_1|-1}{2} \rceil \geq 4^k/2$. Let x_2 be an arbitrary vertex in X_2. Let X_3 be the largest monochromatic neighborhood of x_2 in X_2 with respective edges of color c_2, $x_3 \in X_3$, and so on let X_m be the largest monochromatic neighborhood of x_{m-1} with respective color c_{m-1} in X_{m-1}, $x_m \in X_m$. We see that $|X_m| \geq 4^k/2^{m-1}$. Thus $|X_m| > 0$ as long as $2k > (m-1)$, i.e., as long as $m \leq 2k$. Consider vertices $x_1, \dots, x_{2k}$ and colors $c_1, \dots, c_{2k-1}$. At least k of the colors, say $c_{i1}, c_{i2}, \dots, c_{ik}$ are the same by pigeonhole principle, say without loss of generality, red. Then $x_{i1}, x_{i2}, \dots, x_{ik}$ induce a k-vertex clique all of whose edges are red.



For the lower bound, we shall construct a coloring of K_n, $n = 2^{k/2}$ with no monochromatic cliques on k vertices. Let's color each edge red with probability 1/2 and blue with probability 1/2. Let S be a fixed set of k vertices. Then

$$\text{Prob}(S \text{ induces a red clique}) = 2^{-\binom{k}{2}}.$$

Thus Prob(S induces monochromatic clique) $= 2^{-\binom{k}{2}+1}$. Therefore

$$\begin{aligned} \text{Prob(there is a monochromatic clique on } k \text{ vertices)} &\leq \binom{n}{k} 2^{-\binom{k}{2}+1} \\ &\leq \frac{n^k}{k!} 2^{-k^2/2+k/2+1} \\ &\leq \frac{2^{k/2+1}}{k!} \\ &< 1. \end{aligned}$$

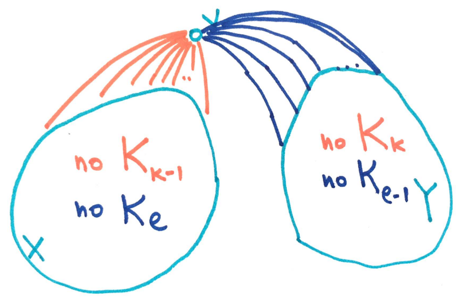
Thus there is a coloring with no monochromatic cliques of size k.

$\square$

Remark. $R(2) = 2$, $R(3) = 6$, $R(4) = 18$ and $43 \leq R(5) \leq 48$.

Theorem 80. For any integers $k, \ell \geq 2$, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

Proof. We shall prove the statement by induction on $k + \ell$ with basis case $k = 2, \ell = 2$. We know that $R(2, 2) = 2 \leq \binom{2+2-2}{1} = 2$.
Consider $R(k, \ell)$. Assume that $R(k', \ell') \leq \binom{k'+\ell'-2}{k'-1}$ if $k' + \ell' < k + \ell$.
Let $N = R(k, \ell) - 1$ and let c be an edge-coloring of $G = K_N$ in red (r) and blue (b) with no red K_k and no blue K_ℓ. Fix a vertex v. Let X and Y be vertex sets such that $X = \{x : c(xv) = r\}$, $Y = \{y : c(yv) = b\}$. Then $G[X]$ does not contain a red K_{k-1} (otherwise together with v it would form a red K_k), and it does not contain a blue K_ℓ. Similarly, $G[Y]$ does not contain a blue $K_{\ell-1}$ and does not contain a red K_k.



By definition of Ramsey number, $|X| \leq R(k-1,\ell)-1$ and $|Y| \leq R(k,\ell-1)-1$. Thus

$$N = |X| + |Y| + 1 \leq R(k-1,\ell) - 1 + R(k,\ell-1) - 1 + 1.$$

On the other hand

$$N = R(k,\ell) - 1.$$

Thus

$$R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1).$$

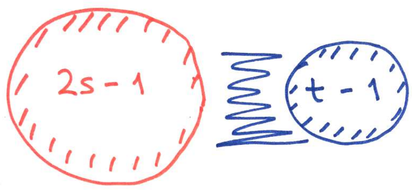
By induction hypothesis, we have

$$R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1) \leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} = \binom{k+\ell-2}{k-1}.$$

□

Theorem 81. Let $s \geq t \geq 1$, $s,t \in \mathbb{Z}$. Then $R(sK_2, tK_2) = 2s + t - 1$.

Proof. Lower bound:
Consider a complete graph G on $2s+t-2$ vertices. Color all edges of a complete subgraph on $2s-1$ vertices red and all remaining edges blue.



Then we see that there is no red sK_2 since the red edges of G span $2s-1$ vertices and sK_2 spans $2s$ vertices. In addition, there is no blue tK_2 because every blue edge in G is incident to a set S of $t-1$ vertices and the smallest number of vertices intersecting all edges of tK_2 is t. This shows that $R(sK_2, tK_2) \geq 2s + t - 1$.

Upper bound:
Next we shall show, by induction on $\min\{s,t\}$, that in any edge coloring of K_{2s+t-1} there is a red sK_2 or there is a blue tK_2. If $t = 1$, then $R(sK_2, K_2) = 2s = 2s + t - 1$. Indeed, if K_{2s} has only red edges, there is a red sK_2. If it has at least one blue edge, there is a blue K_2.

Now let $t \geq 2$ and consider $G = K_{2s+t-1}$ edge-colored red and blue. If all edges of G are red or all edges of G are blue, we have a red sK_2 or a blue tK_2. Thus there are red and blue edges and there are two adjacent edges xy and yz of different colors, say xy is red and yz is blue. Let x, y, z be the vertices in these edges. Since $|V(G) - \{x,y,z\}| = 2s + t - 1 - 3 = 2(s-1) + (t-1) - 1$, we have by induction that

$G - \{x, y, z\}$ contains a red $(s-1)K_2$ or a blue $(t-1)K_2$. Together with xy or yz we have a red sK_2 or a blue tK_2. $\square$

Explicit Ramsey construction

Let $\mathcal{F}$ be a family of k-element subsets of an n-element set. By the result of Ray-Chaudhuri and Wilson,

$$\text{if} \quad |\{|F \cap F'| : F, F' \in \mathcal{F}\}| \leq s, \text{ then } |\mathcal{F}| \leq \binom{n}{s}. \tag{3}$$

By the result of Frankl and Wilson,

$$\text{if} \quad |F \cap F'| \not\equiv k \pmod{q} \text{ for a prime power } q \text{ then } |\mathcal{F}| \leq \binom{n}{q-1}. \tag{4}$$

Theorem 82 (Frankl and Wilson)**.** When k is sufficiently large,

$$r(k) \geq \exp\left(\frac{\log^2 k}{20 \log\log k}\right).$$

Proof. Let $V(G) = \binom{X}{q^2-1}$, where $|X| = q^3$ and q is a sufficiently large prime power. Let

$$E(G) = \{\{F, F'\} : |F \cap F'| \not\equiv -1 \pmod{q}\}.$$

If $F_1, \ldots, F_m$ form a complete graph, then $m \leq \binom{q^3}{q-1}$ by (4). If $F_1, \ldots, F_m$ form an independent set, then the pairwise intersections have sizes $q-1, 2q-1, \ldots, q^2-q-1$, so $m \leq \binom{q^3}{q-1}$ by (3). So, G has no clique or co-clique on k vertices, where

$$|V(G)| = \binom{q^3}{q^2-1} \quad \text{and} \quad k = \binom{q^3}{q-1}.$$

Using the bounds $\left(\frac{n}{m}\right)^m \leq \binom{n}{m} \leq n^m$, we have that

$$q^q \leq k \leq q^{3q} \text{ and } |V(G)| \geq q^{q^2/2}.$$

So $q \log q \leq \log k \leq 3q \log q$ and thus $\log k / 3 \log q \leq q \leq \log k / \log q$. Therefore $\log q \leq \log\log k - \log\log q \leq \log\log k$ and thus $q \geq \log k / 3 \log\log k$. Therefore

$$\begin{aligned} |V(G)| &\geq (\log k / 3 \log\log k)^{\log^2 k / 18 (\log\log k)^2} \\ &= \exp(\log^2 k (\log\log k - \log 3 - \log\log\log k) / 18 (\log\log k)^2) \\ &\geq \exp(\log^2 k / 20 \log\log k). \end{aligned}$$

$\square$

Note that this gives that $r(k) \geq k^{c\sqrt{\log k}}$, i.e., this bound is greater than any power of k but smaller than exponential. The best constructive bound up to date is due to Barak, Rao, Shatiel and Wigderson: $r(k) \geq \exp((1+o(1))\log^{(2+a)k})$, for a positive constant a.

Let $R(p, q; r)$ be the hypergraph Ramsey number for r-uniform hypergraphs, i.e.,

$$\begin{aligned} R(p,q;r) = \min \Bigg\{ N : \ \forall c : \binom{[N]}{r} \to \{0,1\} \\ \exists A \subseteq [N], |A| = p, \forall A' \in \binom{A}{r} \ c(A') = 0 \text{ or} \\ \exists B \subseteq [N], |B| = q, \forall B' \in \binom{B}{r} \ c(B') = 1 \Bigg\} \end{aligned}$$

We say that a complete r-uniform hypergraph on n vertices is an r-clique on n vertices.

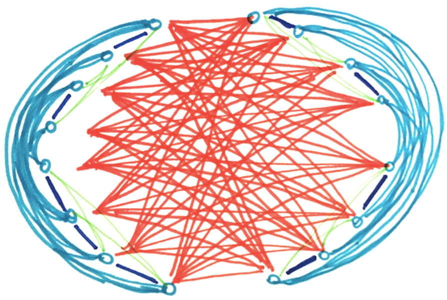
The following theorem show the existence of hypergraph Ramsey numbers.

Theorem 83. For any parameters $p, q, r \geq 2$,

$$R(p, q; r) \leq R\big(R(p-1, q; r),\ R(p, q-1; r);\ r-1\big) + 1.$$

Proof. Let $c : \binom{X}{r} \to \{red, blue\}$, where $|X| = R(R(p-1, q; r), R(p, q-1; r); r-1)+1$. We shall show that there is a red r-clique on p vertices or a blue r-clique on q vertices. Let $x \in X$. Let $c' : \binom{X-x}{r-1} \to \{red, blue\}$ be defined as follows: for any $A \subseteq X - x$, let $c'(A) = c(A \cup x)$. Let $p_1 = R(p-1, q; r)$ and $q_1 = R(p, q-1; r)$. Since $|X - x| = R(p_1, q_1; r-1)$, there is a red $(r-1)$-clique on vertex set X', $|X'| = p_1$, or a blue $(r-1)$-clique on vertex set X'', $|X''| = q_1$. Assume the former. The latter is treated similarly. Then in c, all sets $A \cup x$ are red, where $A \subseteq X'$. Since $|X'| = p_1 = R(p-1, q; r)$, then in X' under c there is either a blue r-clique of size q and we are done, or there is a red r-clique on vertex set $X^* \subseteq X'$, $|X^*| = p - 1$. But then $X^* \cup x$ forms a red r-clique under c on p vertices and we are done. $\square$

Lemma 84. We have $c_1 \cdot 2^k \leq R_2(\underbrace{3, \ldots, 3}_{k}) \leq c_2 \cdot k!$ for some constants $c_1, c_2 > 0$.



Applications of Ramsey theory

Theorem 85 (Erdős, Szekeres)**.** Any list of more than n^2 numbers contains a non-decresing or non-increasing sublist of more than n numbers.

Proof. Let $a_1, \ldots, a_{n^2+1}$ be a list of numbers. Let u_i be the length of a longest non-decreasing sublist ending with a_i. Let d_i be the length of a longest non-increasing sublist ending with a_i. Assume that the statement of the theorem is false. Then $u_i, d_i \leq n$ and there are at most n^2 distinct pairs (u_i, d_i). Since there are more than n^2 numbers there are indices $i < j$ such that $(u_i, d_i) = (u_j, d_j)$. If $a_i \leq a_j$, we have $u_i < u_j$. If $a_i \geq a_j$, we have $d_i < d_j$, a contradiction. $\square$

Theorem 86 (Erdős, Szekeres)**.** For any integer m, $m \geq 3$, there is an integer $N = N(m)$ such that if X is a set of N points on the plane such that no three points are on a line, then X contains a vertex set of a convex m-gon.

Proof. Let $N = R(m, 5; 4)$. For each 4-element subset X' of X color it red if the convex hull of X' is a 4-gon, it blue if the convex hull of X' is a triangle. By definition of R, there is either a set A of m points, such that $\binom{A}{4}$ is red, or a set B of 5 points such that $\binom{B}{4}$ is blue. Assume the latter. Then we see in particular that the convex hull of B is a triangle T and there are two vertices u, v of B inside this triangle. Consider a line through u, v, it splits the plane in two parts, one containing one vertex of T, another two vertices of T, call them x, y. Then the convex hull of $\{u, v, x, y\}$ is a 4-gon, so $\{x, y, u, v\}$ is colored red, a contradiction. Therefore there is a set A of m points, such that $\binom{A}{4}$ is red. We claim that A forms a vertex set of a convex m-gon. Assume not, and there is a point x of A inside the convex hull A' of A. Triangulate A'. Then x will be inside one of the triangles, say with vertex set $\{y, z, w\}$. Then $\{x, y, z, w\}$ must be colored blue, a contradiction. $\square$

Theorem 87 (Schur)**.** For any number of colors k, there is a large enough $N \in \mathbb{N}$ so that and any coloring of $\{1, 2, \ldots, N\}$ with k colors, there are numbers x, y, and z of the same color such that $x + y = z$.

Proof. Let $N = R_k(3, 3, \ldots, 3)$, where R is the multicolor Ramsey number for graphs with k colors. Let $c : [N] \to [k]$. Let $c' : E(K_N) \to [k]$ so that $V(K_N) = [N]$ and $c'(ij) = c(|i - j|)$. Then by Ramsey theorem, there is a monochromatic triangle i, j, l, $i < j < l$, in K_N, say of color s. So, $c(l - j) = c(j - i) = c(l - i) = s$. Let $x = l - j, y = j - i, z = l - i$. Then $x + y = z$ and $c(x) = c(y) = c(z) = s$. $\square$

Definition 7.3. Let $r \in \mathbb{N}$ and $A \in \mathbb{Z}^{n \times k}$.

- Matrix A is said to be r-*regular* if there is a monochromatic solution of $Ax = 0$ for any r-coloring $c \colon \mathbb{N} \to [r]$ of $\mathbb{N}$. r-regular matrix

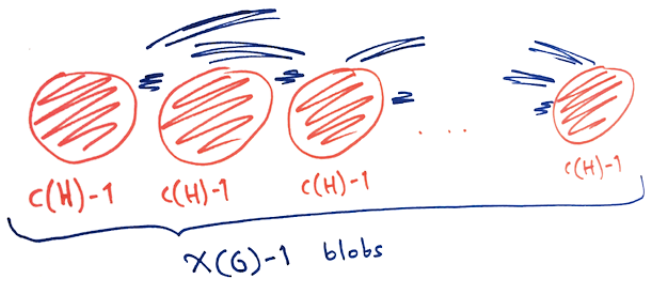
- Matrix A fulfils the *column condition* if there is a partition $C_1 \dot{\cup} \cdots \dot{\cup} C_l$ of the columns of A such that the following holds: Let $s_i := \sum_{c \in C_i} c$ for $i \in [l]$ be the sum of columns in C_i. Then $s_1 = 0$ and every s_i for $i \in \{2, \ldots, l\}$ is a linear combination of the columns in $C_1 \dot{\cup} \ldots \dot{\cup} C_{i-1}$. column condition

  For example, $2x_1 + x_2 + x_3 - 4x_4$ fulfils the column condition since $2+1+1-4=0$.

Theorem 88 (Rado)**.** Let $A \in \mathbb{Z}^{n \times k}$. If A fulfils the column condition, then A is r-regular for every $r \in \mathbb{N}$.

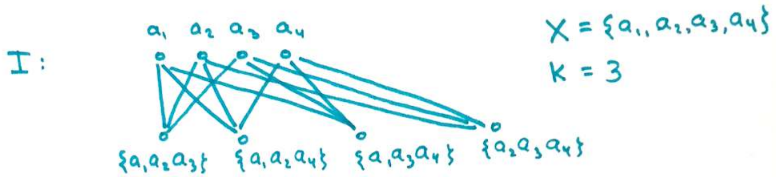
Lemma 89. For any $s, t \in \mathbb{N}$ with $s \geq t \geq 1$ and $s \geq 2$ we have $R(sK_3, tK_3) = 3s + 2t$.

Theorem 90 (Chvátal, Harary)**.** Let G and H be graphs. Then $R(G, H) \geq \big(\chi(G) - 1\big)\big(c(H) - 1\big) + 1$ where $c(H)$ is the order of the largest component of H.



Induced Ramsey numbers

We say that $G \underset{ind}{\rightarrow} H$ if in any coloring of $E(G)$ there is a monochromatic induced copy of H. We shall prove bipartite induced Ramsey theorem. We need two lemmas for that. We say that for a set X and a positive integer $k \leq |X|$, a bipartite graph $(X \cup \binom{X}{k}, E)$ is an *incidence graph* if $E = \{X'x : X' \in \binom{X}{k}, x \in X, x \in X'\}$.

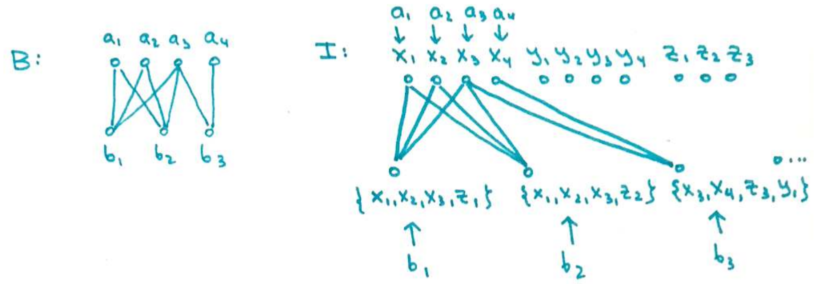


Lemma 91. For any bipartite graph B, there is an incidence graph containing B as an induced subgraph.

Proof. Let $B = (\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}, E)$. Let I be an incidence graph, $I = (X \cup \binom{X}{n+1}, E)$, where $X = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \cup \{z_1, \ldots, z_m\}$. Let

$\phi : V(B) \to V(I)$ defined as follows:

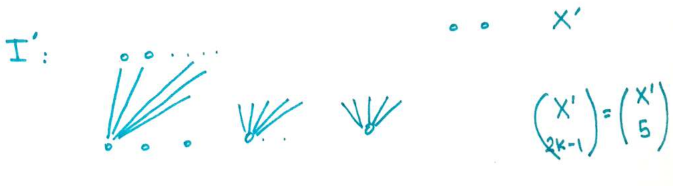
$$\phi(a_i) = x_i,\ i = 1, \ldots, n,$$
$$\phi(b_i) = \{z_i\} \cup \{\phi(v) : v \in N_B(b_i)\} \cup Y_i,\ Y_i \subseteq \{y_1, \ldots, y_n\},\ i = 1, \ldots, n.$$



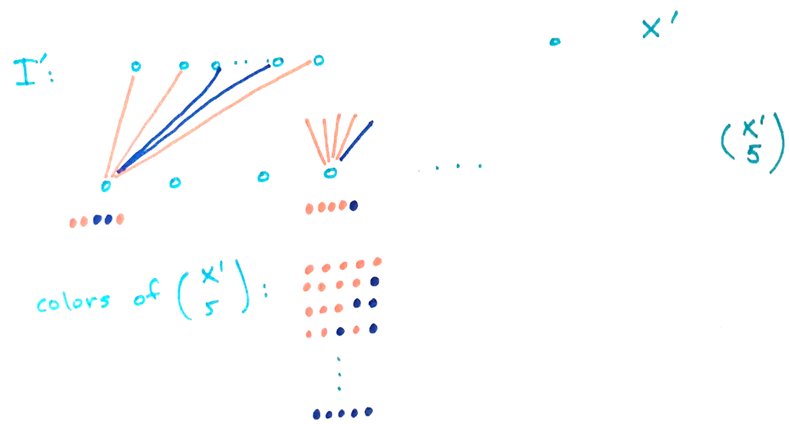
We see that since for distinct B_i's, $\phi(b_i)$ contain distinct z_i's, all vertices $b_1, \ldots, b_n$ are mapped into n distinct vertices of $\binom{X}{n+1}$. Morever, we see that $\phi(a_i)\phi(b_j) \in E(I)$ if and only if $a_i b_j \in E(B)$. Indeed, if $a_i b_j \in E(B)$, then $\phi(a_i) = x_i$ and, as $a_i \in N_B(b_j)$, $x_i \in \phi(b_j)$, thus $\phi(a_i)\phi(b_j) \in E(I)$. The other way around, if $\phi(a_i)\phi(b_j) \in E(I)$, then as $\phi(a_i) = x_i$, $x_i \in \phi(b_j)$, thus $a_i \in N_B(b_j)$, so $a_i b_j \in E(B)$. This shows that B is an induced subgraph of I. □

Lemma 92. Let $I = (X \cup \binom{X}{k}, E)$ and $I' = (X' \cup \binom{X'}{2k-1}, E')$, be two incidence graphs, with $|X'|$ corresponding to the multicolor hypergraph Ramsey number with 2^{2k-1} colors, uniformity $2k-1$, and the order of unavoidable monochromatic clique is $k|X| + k - 1$. Then $I' \underset{ind}{\to} I$.

Proof. Fix an order on $X' = \{x_1, x_2, \ldots\}$. Let $Y' = \binom{X'}{2k-1}$. Let $c : E' \to \{r, b\}$, red and blue. I.e., c is an edge coloring of the bipartite graph I' with parts X', Y'. Each vertex in Y' is incident to $2k-1$ edges colored r or b.

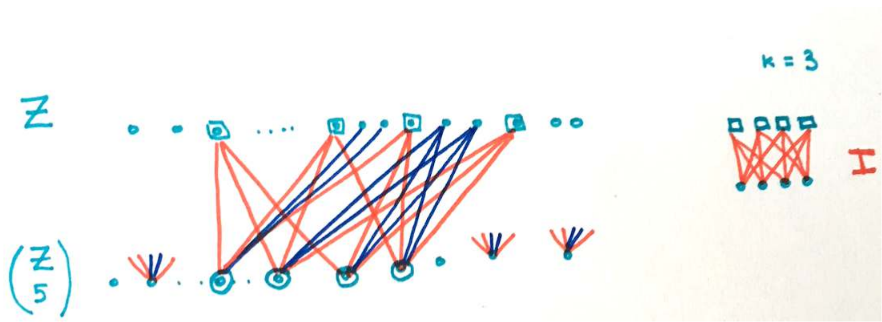


On the other hand, the vertices of Y' correspond to hyperedges of a complete $(2k-1)$-uniform hypergraph H on vertex set X'. Let c' be the coloring of hyperedges of H such that $c'(y') = (c(y'x_{i_1}), c(y'x_{i_2}), \ldots, c(y'x_{i_{2k-1}}))$, $i_1 < i_2 < \ldots < i_{2k-1}$. I.e., c' assigns binary vectors of lengths $2k-1$ to the hyperedges of H. Thus the total number of colors is at most 2^{2k-1}.



Since $|X'|$ was defined as a respective Ramsey number, there is a set $Z \subseteq X'$, $|Z| = k|X| + k - 1$, such each set in $\binom{Z}{2k-1}$ has the same color. This color is a vector with entries r or b with $2k-1$ entries. By pigeonhole principle at least k of these entries are the same, without loss of generality, r. Call the coordinates of the k red entries *good*.

Now we shall find a red copy of I in a subgraph of I induced by Z and its neighbors. We shall provide an explicit embedding ϕ of I. Let vertices $\phi(x)$, $x \in X$ be in Z such that there are $k-1$ vertices of Z between consecutive $\phi(x)$'s. Let $Z' = Z - \{\phi(x) : x \in X\}$. Let $y \in \binom{X}{k}$, say $y = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$. Then let $\phi(y) = \{\phi(x_{i_1}), \dots, \phi(x_{i_k})\} \cup \{Z''\}$, where $Z'' \subseteq Z'$ and $\phi(x_{i_1}), \dots, \phi(x_{i_k})$ occupy good positions in $\phi(y)$. We see that ϕ maps vertices of I into an induced subgraph of I' isomorphic to I. Moreover, since all edges corresponding to good positions are red, this subgraph is red.



□

Theorem 93. For any bipartite graph B there is a bipartite graph I' such that $I' \underset{ind}{\to} B$.

Proof. By the first lemma, we see that there is an incidence graph I such that B is an induced subgraph of I. Let I' be an incidence graph guaranteed by the second lemma, such that $I' \underset{ind}{\to} I$. Then $I' \underset{ind}{\to} B$. □

Theorem 94 (Induced Ramsey Theorem, Deuber, Erdős, Hajnal & Pósa, 9.3.1)**.** We have that $IR(G, H)$ is finite for all graphs G and H.

Concerning upper bounds for induced Ramsey numbers, there is the following conjecture due to Erdős: if G is an n-vertex graph, then there is a constant $c > 0$ such that

$$IR(G, G) \leq 2^{cn}.$$

The best known upper bound is due to Conlon, Fox, and Sudakov, who showed that $IR(G, G) \leq 2^{cn \log n}$.

We say that a complete graph is lexically edge colored with a coloring c if its vertices can be ordered $v_1, \ldots, v_n$ such that $c(v_i v_j) = c(v_i v_k)$ for all $i < j < k$ and moreover $c(v_i v_{i+1}) \neq c(v_j v_{j+1})$ for any $1 \leq i < j < n$.

Theorem 95 (Canonical Ramsey Theorem, Erdős-Rado 1950)**.** For all $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that any edge coloring of K_n with arbitrarily many colors contains a K_k that is monochromatic, rainbow or lexical.

Theorem 7.4 (Chvátal-Rödl-Szemerédi-Trotter, 9.2.2)**.** For any positive integer Δ there exists a $c \in \mathbb{N}$ such that for every graph H with $\Delta(H) = \Delta$ we have $R(H, H) \leq c|V(H)|$.

Corollary 96. For any n-vertex graph H with maximum degree 3 we have $R(H, H) \leq cn$ for some constant $c > 0$. This number grows much slower than $R(K_n, K_n) \geq \sqrt{2}^n$.

In 1973, Burr and Erdős conjectured that for every positive integer d there is a constant $c = c(d)$ such that if H is a d-degenerate graph, then $R(H, H) \leq c|V(H)|$. This was established in 2016 by Choongbum Lee:

Theorem 97 (Lee)**.** For every natural number d there is a constant $c = c(d)$ such that if H is d-degenerate graph on n vertices, then $R(H, H) \leq cn$.

Theorem 98 (Anti-Ramsey Theorem, Erdős-Simonovits-Sós)**.** For all $n, r \in \mathbb{N}$ we have $AR(n, K_r) = \binom{n}{2}\big(1 - 1/(r-2)\big)\big(1 - o(1)\big)$.

8 Flows

Let H be an abelian group and G be a multigraph.

Note that some definitions of a multigraph are done using multisets. We can avoid using multisets by considering the following definition. A *multigraph* is a triple, (V, E, T^*), where V and E are sets called vertex set and edge set respectively, and T^* is a set of tuples $(\{x, y\}, e)$, $x, y \in V(G)$, $e \in E$, such that for each $e \in E$ there is a unique tuple $(\{x, y\}, e)$ in T^*. If $(\{x, y\}, e) \in T^*$, we say that x and y are endpoints of e, if $x = y$, we say that e is a loop. If $(\{x, y\}, e), (\{x, y\}, e') \in T^*$, $e \neq e'$, $x \neq y$, we say that e and e' are parallel or multiple edges.

For an edge e with endpoints x and y, we shall be assigning a value to an ordered triple (x, e, y). Let $T(G) = \{(x, e, y) : (\{x, y\}, e) \in T^*(G)\}$.

Let $f : T(G) \to H$ and let $X, Y \subseteq V(G)$. We define

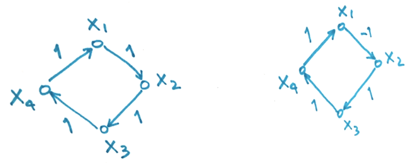
$$f(X, Y) := \sum_{x \in X, y \in Y, (x,e,y) \in T(G), x \neq y} f(x, e, y),$$

and write $f(x, Y)$ for $f(\{x\}, Y)$.

Definition 8.1. We say that a map $f : T(G) \to H$, is a *circulation on* G if circulation

(F1) $f(x, e, y) = -f(y, e, x)$ for any edge with endpoints x and y, $x \neq y$

(F2) $f(x, V(G)) = 0$.



Lemma 99. Let f be a circulation on a multigraph G. Then for any subset X of vertices

- $f(X, X) = 0$,
- $f(X, V(G)) = 0$,
- $f(X, V(G) - X) = 0$,
- If $e = xy$ is a bridge, then $f(x, e, y) = 0$.

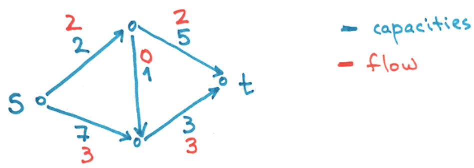
Let G be a graph, $s, t \in V(G)$, be distinct vertices and $c : T(G) \to \mathbb{N} \cup \{0\}$. We call a quadruple $N = (G, s, t, c)$ a *network* with *source* s, *sink* t and *capacity function* c. network, source, sink, capacity

Definition 8.2. A function $f : T(G) \to \mathbb{R}$ is a *network flow*, or *N-flow* if network flow

(F1) $f(x, e, y) = -f(y, e, x)$ for any edge e with endpoints x and y, $x \neq y$,

(F2) $f(x, V(G)) = 0$, $x \in V(G) - \{s, t\}$ and

(F3) $f(x, e, y) \leq c(x, e, y)$ for any edge e with endpoints x and y, $x \neq y$.



A *cut* in a network N is a pair $(S, \overline{S})$, where S is a subset of vertices of G such that $s \in S$, $t \notin S$ and $\overline{S} = V(G) - S$. cut

We say that $c(S, \overline{S}) = \sum_{x \in S, y \in \overline{S}, (x,e,y) \in T(G)} c(x, y)$ is the *capacity* of the cut. capacity of cut

Lemma 100. For any cut $(S, \overline{S})$ and a network flow f in a network N, $f(S, \overline{S}) = f(s, V(G))$.

Proof.

$$f(S, \overline{S}) = f(S, V) - f(S, S) = f(s, V) + \sum_{v \in S \setminus \{s\}} f(v, V) - f(S, S) = f(s, V) + 0 - 0,$$

where we use properties (F1) and (F2). □

Thus $f(S, \overline{S})$ does not depend on the cut. The value $f(s, V)$ is also called the *value of f* and is denoted by $|f|$. value, $|f|$

Theorem 8.3 (Ford-Fulkerson Theorem, 6.2.2)**.** Let $N = (G, s, t, c)$ be a network. Then

$$\max\{|f| : \ f \text{ is an } N- \text{ flow } \} = \min\{c(S, \overline{S}) : \ (S, \overline{S}) \text{ is a cut}\},$$

and there is an integral flow $f : T \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

Proof. Since $|f| = f(s, V) = f(S, \overline{S}) \leq c(S, \overline{S})$, for any cut $(S, \overline{S})$, we have that

$$\max\{|f| : \ f \text{ is an } N- \text{ flow } \} \leq \min\{c(S, \overline{S}) : \ (S, \overline{S}) \text{ is a cut}\}.$$

Next, we shall construct a flow f such that $|f| = \min\{c(S, \overline{S}) : \ (S, \overline{S}) \text{ is a cut}\}$.

We shall define $f_0, f_1, \ldots$ - a sequence of N-flows such that $f_0(x,e,y) = 0$ for all $(x,e,y) \in T(G)$, f_i assigns integer values and $|f_i| \geq |f_{i-1}| + 1$ for $i \geq 1$. Note that since $|f_i| \leq \min\{c(S,\overline{S}) : (S,\overline{S}) \text{ is a cut}\}$ for all $i = 0, 1, \ldots,$ the sequence $f_0, f_1, \ldots$ is finite. Let f_n be defined. We shall either let $f = f_n$ or define f_{n+1}.

Case 1 There is a sequence of vertices $x_0 = s, x_1, \ldots, x_m = t$ and edges $e_0, \ldots, e_{m-1}$ such that $x_i x_{i+1} = e_i \in E(G)$ and $f(x_i, e_i, x_{i+1}) < c(x_i, e_i, x_{i+1})$, $i = 0, \ldots, m-1$.

Let $\epsilon = \min\{c(x_i, e_i, x_{i+1}) - f(x_i, e_i, x_{i+1}) : i = 0, \ldots, m-1\}$. Note that $\epsilon \in \mathbb{N}$. Let

$$f_{n+1}(x,e,y) = \begin{cases} f_n(x,e,y), & (x,e,y) \neq (x_i, e_i, x_{i+1}), i = 0, \ldots, m-1, \\ f_n(x,e,y) + \epsilon, & (x,e,y) = (x_i, e_i x_{i+1}), i = 0, \ldots, m-1, \\ f_n(x,e,y) - \epsilon, & (x,e,y) = (x_{i+1}, e_i, x_i), i = 0, \ldots, m-1. \end{cases}$$

Note that f_{n+1} is an N-flow, it takes integer values, and $|f_{n+1}| = |f_n| + \epsilon \geq |f_n| + 1$.

Case 2 Case 1 does not hold. Let

$$S = \{v \in V : \exists \text{ path } s = x_0, e_0, x_1, \ldots, e_q, x_{q+1} = v,$$

$$f(x_i, e_i, x_{i+1}) < c(x_i, e_i, x_{i+1}), i = 0, \ldots, q\}.$$

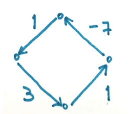
Note that since we are not in Case 1, $t \notin S$. Also, $s \in S$. Thus $(S, \overline{S})$ is a cut. From the definition of S, we see that $f_n(x,e,y) = c(x,e,y)$ for all $x \in S, y \in \overline{S}$, $(x,e,y) \in T(G)$. Thus $f_n(S, \overline{S}) = c(S, \overline{S})$ and so $|f_n| \geq \min\{c(S,\overline{S}) : (S,\overline{S}) \text{ is a cut}\}$. Let $f = f_n$.

Since the sequence $f_0, f_1, \ldots$ is finite, Case 2 must occur. □

Group-valued flows

Definition 8.4. Let $G = (V, E, T^*)$ be a multigraph.

- If H is an abelian group, then a circulation f is an *H-flow on G* if $f(x,e,y) \neq 0$ for all $(x,e,y) \in T$. Sometimes f referred to as a *nowhere-zero flow*. *H*-flow nowhere-zero



A nowhere-zero $\mathbb{Z}_2$-flow.

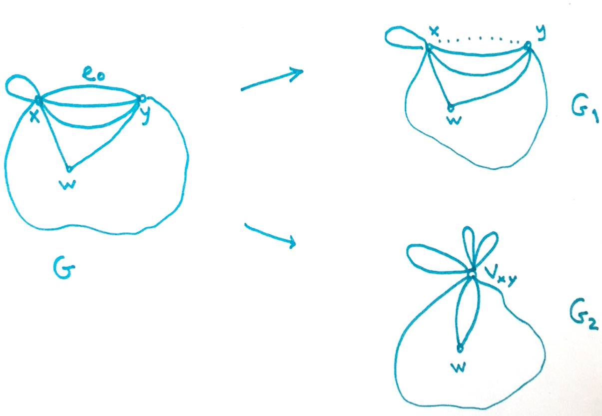
- For $k \in \mathbb{N}$ a *k-flow* is a $\mathbb{Z}$-flow f such that $0 < |f(x,e,y)| < k$ for all $(x,e,y) \in T$. The *flow number* $\varphi(G)$ *of* G is the smallest k such that G has a k-flow. k-flow flow number, $\varphi(G)$

Theorem 8.5 (Tutte, 6.3.1)**.** For every multigraph $G = (V, E, T^*)$ there is a polynomial $P \in \mathbb{Z}[X]$ such that for any finite abelian group H the number of nowhere-zero H-flows on G is $P\big(|H| - 1\big)$.

Proof. Use induction on the number of non-loop edges in G.
If this number is zero, i.e., all edges are loop edges, for any triple (x, e, x) with $(\{x, x\}, e) \in T^*$, one can assign any value from $H - \{0\}$ and obtain an H-flow. The number of such assignments is $(|H| - 1)^{||G||}$, that is a polynomial of $|H| - 1$.

Assume there is a non-loop edge e_0 with endpoints x and y.
Let $G_1 = G - e_0$, $G_2 = G/e_0$, where G/e_0 is a graph obtained from G by contracting the endpoints x, y into a vertex $v = v_{xy}$ of e_0 and removing the obtained loop (v, e_0, v). More formally,

$$
\begin{aligned}
V(G/e_0) &= V(G) - \{x, y\} \cup \{v\}, \\
E(G/e_0) &= E(G) - \{e_0\}, \\
T^*(G/e_0) &= \big(T^*(G) \setminus \{(\{w, w'\}, e) : w \in \{x, y\}, e \in E(G)\}\big) \\
&\quad \cup \big\{(\{v, w\}, e) : (\{x, w\}, e) \in T^*(G) \text{ or} (\{y, w\}, e) \in T^*(G) \text{ and } e \neq e_0\big\}.
\end{aligned}
$$



We define the following sets:

$$
\begin{aligned}
F &:= \{f : \ f \text{ is an } H\text{-flow on } G\}, \\
F_1' &:= \{f : \ f \text{ is an } H\text{-flow on } G_1\}, \\
F_2' &:= \{f : \ f \text{ is an } H\text{-flow on } G_2\},
\end{aligned}
$$

$$
\begin{aligned}
F_1 &:= \{f : \ f \text{ is an } H\text{-circulation on } G \text{ such that } f(x, e, y) = 0 \text{ iff } e = e_0\}, \\
F_2 &:= F_1 \cup F.
\end{aligned}
$$

Let P_1 and P_2 are polynomials guaranteed by the induction hypothesis with respect to G_1 and G_2, i.e., $|F_i'| = P_i(|H|-1)$, $i = 1, 2$. We shall prove that the number of H-flows of G is $P_2(|H|-1) - P_1(|H|-1)$, thus a polynomial of $|H|-1$.

It is sufficient for us to show that $|F_2'| = |F_2|$ and $|F_1'| = |F_1|$. Indeed, if this is the case, then, we have that $|F_1'| = P_1(|H|-1)$, $|F_2'| = P_2(|H|-1)$, so $|F| = |F_2| - |F_1| = |F_2'| - |F_1'| = P_2(|H|-1) - P_1(|H|-1)$, as desired.

It is easy to see that $|F_1| = |F_1'|$. To see that $|F_2| = |F_2'|$, we shall build two injections: $\lambda : F_2 \to F_2'$ and $\mu : F_2' \to F_2$.

Let $f \in F_2$. Note that $f(\{x, y\}, V - \{x, y\}) = 0$ by Lemma 99. Construct $g = \lambda(f) \in F_2$ as $g(x', e, y') = f(x', e, y')$ for all $(x', e, y') \in T(G) \cap T(G_1)$, $g(x, e, t) = f(v, e, t)$ for all $(x, e, t) \in T(G)$ with $t \neq v$, $g(y, e, t) = f(v, e, t)$ for all $(y, e, t) \in T(G)$ with $t \neq v$ and $g(t, e, v) = -g(v, e, t)$ for all $(t, e, v) \in T(G_2)$ with $v \neq t$. Finally, let $g(v, e, v)$ be equal to $f(x, e, x)$ if $(x, e, x) \in T(G)$, $f(y, e, y)$ if $(y, e, y) \in T(G)$ or $f(x, e, y)$ if $(x, e, y) \in T(G), e \neq e_0$.

We see that g is an H-flow since (F1) is satisfied and (F2) is satisfied at v because of $f(\{x, y\}, V - \{x, y\}) = 0$ and (F2) is satisfied for other vertices because the values of g on respective edges are the same as in f. To see that λ is an injection, consider two different elements $f_1, f_2 \in F_2$. If they differ on the triple (x', e, y'), the images $\lambda(f_1)$ and $\lambda(f_2)$ differ on the triple containing e.

Let $f \in F_2'$, let us construct $g = \mu(f) \in F_2$. Let $g(w_1, e, w_2) = f(w_1', e, w_2')$ for all $(w_1, e, w_2) \in T(G_2)$ and $(w_1', e, w_2') \in T(G)$, where $e \neq e_0$.

Construct $g = \mu(f) \in F_2'$ as $g(x', e, y') = f(x', e, y')$ for all $(x', e, y') \in T(G) \cap T(G_2)$, $g(v, e, t) = f(x, e, t)$ for all $(x, e, t) \in T(G)$ with $t \neq v$, $g(v, e, t) = f(y, e, t)$ for all $(y, e, t) \in T(G)$ with $t \neq v$, $g(t, e, u) = -g(u, e, t)$ for all $(t, e, v) \in T(G_2)$ with $v \neq t$ and $u \in \{x, y\}$ and $g(x, e, y) = f(v, e, v)$, $g(y, e, x) = -f(v, e, v)$ for all $(x, e, y) \in T(G)$. Finally, let $g(x, e, x) = f(v, e, v)$ for all $(x, e, x) \in T(G)$ and $g(y, e, y) = f(v, e, v)$ for all $(y, e, y) \in T(G)$.

Moreover, let

$$\begin{aligned} g(x, e_0, y) &= -g(x, V - \{x, y\}) - \sum_{(x,e,y) \in T(G), x \neq y, e \neq e_0} g(x, e, y) \text{ and} \\ g(y, e_0, x) &= -g(y, V - \{x, y\}) - \sum_{(x,e,y) \in T(G), x \neq y, e \neq e_0} g(y, e, x). \end{aligned}$$

Note that these values could be zero.

We need to check that g is a circulation that is non-zero on triples involving all edges except perhaps e_0. Property (F1) is satisfied by construction on $e \neq e_0$. To see that

(F1) is satisfied on e_0, recall that $g(x, V-\{x,y\}) + g(y, V-\{x,y\}) = g(\{x,y\}, V-\{x,y\}) = f(v, V-v) = 0$ and $g(x,e,y) = -g(y,e,x)$. So, $g(x, V-\{x,y\}) = -g(y, V-\{x,y\})$ and $\sum_{(x,e,y)\in T(G), e\neq e_0, x\neq y} g(x,e,y) = -\sum_{(x,e,y)\in T(G), e\neq e_0, x\neq y} g(y,e,x)$. Thus

$$g(x, e_0, y) = -g(y, e_0, x).$$

To ensure (F2), we need that $g(w, V) = 0$, for all $w \in V$. If $w \notin \{x,y\}$, it holds from construction and the fact that $f(w, V) = 0$. If $w \in \{x,y\}$, we have

$$\begin{aligned}
g(x,V) &= g(x, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y} g(x,e,y) \\
&= g(x, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(x,e,y) + g(x,e_0,y), \\
g(y,V) &= g(y, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y} g(y,e,x) \\
&= g(y, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(y,e,x) + g(y,e_0,x).
\end{aligned}$$

Plug these values for $g(x,e_0,y)$ and $g(y,e_0,x)$ to obtain

$$\begin{aligned}
g(x,V) &= g(x, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(x,e,y) \\
&\quad +(-g(x, V-\{x,y\}) - \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(x,e,y)), \\
g(y,V) &= g(y, V-\{x,y\}) + \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(y,e,x) \\
&\quad +(-g(y, V-\{x,y\}) - \sum_{(x,e,y)\in T(G), x\neq y, e\neq e_0} g(y,e,x)).
\end{aligned}$$

Thus $g(x,V) = g(y,V) = 0$. So, g is a desired circulation.

We only need to check that μ is an injection. Consider two distinct maps $f_1, f_2 \in F_2'$. If they differ on a triple (x', e, y') such that either x' or y' is not equal to v, then the respective maps $\mu(f_1)$ and $\mu(f_2)$ are distinct. If $f_1(v,e,v) \neq f_2(v,e,v)$ for $e \neq e_0$, then $\mu(f_1) \neq \mu(f_2)$ on the respective triple. Finally, if f_1 and f_2 coincide on all triples not involving e_0, then $\mu(f_1(x,e_0,y)) = \mu(f_2(x,e_0,y))$ by definition of the value assigned to (x,e_0,y) (as it was expressed in terms of values on other triples). Thus μ is an injection.

□

Corollary 101. If an H-flow on G exists for some finite Abelian group H, then there is also an $\tilde{H}$-flow on G for all finite Abelian groups $\tilde{H}$ with $|\tilde{H}| = |H|$. For example, if a $\mathbb{Z}_4$-flow exists, then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow also exists.

Theorem 102 (Tutte, 6.3.3)**.** A multigraph admits a k-flow if and only if it admits a $\mathbb{Z}_k$-flow.

Theorem 103 (Tutte, 6.5.3)**.** For a plane graph G and its dual G^* we have $\chi(G) = \varphi(G^*)$.

Lemma 104. A graph has a 2-flow if and only if all of its degrees are even.

Lemma 105. A cubic (3-regular) graph has a 3-flow if and only if it is bipartite.

Conjecture (Tutte's 5-Flow Conjecture)**.** Every bridgeless multigraph has flow number at most 5.

Theorem 106 (Seymour, 6.6.1)**.** Every bridgeless graph has flow number at most 6.

9 Random graphs

In this section we deal with randomly chosen graphs. We will often use the "probabilistic method", a proof method for showing existence: By proving that an object with some desired properties can be chosen randomly (in some probability space) with non-zero probability, we also show that such an object exists.

Definition 9.1.

- $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$. This model is called the *Erdős-Rényi model* of random graphs. Erdős-Rényi
- A *property* $\mathcal{P}$ is a set of graphs, e.g. $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$. property

  Let $(p_n) \in [0,1]^{\mathbb{N}}$ be a sequence. We say that $G \in \mathcal{G}(n,p_n)$ *almost always* has property $\mathcal{P}$ if $\mathrm{Prob}\big(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}\big) \to 1$ for $n \to \infty$. If (p_n) is constant p, we also say in this case that *almost all* graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}$. almost always
- A function $f(n)\colon \mathbb{N} \to [0,1]$ is a *threshold function* for property $\mathcal{P}$ if: threshold function
  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \xrightarrow{n\to\infty} 0$ the graph $G \in \mathcal{G}(n,p_n)$ almost always does not have property $\mathcal{P}$.
  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \xrightarrow{n\to\infty} \infty$ the graph $G \in \mathcal{G}(n,p_n)$ almost always has property $\mathcal{P}$.

  Note that not all properties $\mathcal{P}$ have a threshold function.

Lemma 107. Let $G \in \mathcal{G}(n,p)$, let $S \subseteq V(G)$. Let H be a fixed graph on m edges and vertex set S. Then

$$\mathrm{Prob}(G[S] = H) = p^m(1-p)^{\binom{|S|}{2}-m}, \qquad \mathrm{Prob}(H \subseteq G[S]) = p^m.$$

In particular, for a given graph H on n vertices and m edges,

$$\mathrm{Prob}(H = \mathcal{G}(n,p)) = p^m(1-p)^{\binom{n}{2}-m}.$$

Proof. Since the edges are chosen independently with probability p, we choose the m edges of H with probability p^m and $\binom{|S|}{2} - m$ non-edges of H with probability $(1-p)^{\binom{|S|}{2}-m}$. For subgraph containment, we care only about the edges, and chose or do not choose the other pairs with probability 1 each. $\square$

Lemma 108. Let $p \in (0,1)$ be fixed, suppose $G \in \mathcal{G}(n,p)$, and let H be a fixed graph. Then $\text{Prob}(H \underset{ind}{\subseteq} G) \overset{n\to\infty}{\to} 1$.

Proof. Let $k = |V(H)|$. Let $n = tk + \epsilon$, $0 \le \epsilon < k$. Consider t pairwise disjoint sets $A_1, \ldots, A_t$ of vertices in G, each of size k. Then

$$\begin{aligned}
\text{Prob}(H \underset{ind}{\not\subseteq} G) &\le \text{Prob}(H \underset{ind}{\not\subseteq} G[A_1] \wedge H \underset{ind}{\not\subseteq} G[A_2] \wedge \cdots \wedge H \underset{ind}{\not\subseteq} G[A_t]) \\
&= \text{Prob}(H \underset{ind}{\not\subseteq} G[A_1]) \cdots \text{Prob}(H \underset{ind}{\not\subseteq} G[A_t]) \\
&\le (1-r)^t,
\end{aligned}$$

where r is the probability that a k-vertex subset of G induces an isomorphic copy of H. We see that r depends on k, p, and H, but is does not depend on n. So, it is an absolute constant greater than zero. On the other hand, $t = \lfloor n/k \rfloor \to \infty$ as $n \to \infty$. Thus $\text{Prob}(H \underset{ind}{\not\subseteq} G) \to 0$ as $n \to \infty$. $\square$

Lemma 109. Let $n \ge k \ge 2$ be integers. Let $G \in \mathcal{G}(n,p)$.
Then $\text{Prob}(\alpha(G) \ge k) \le \binom{n}{k} (1-p)^{\binom{k}{2}}$ and $\text{Prob}(\omega(G) \ge k) \le \binom{n}{k} p^{\binom{k}{2}}$.

Proof. The probability that a fixed k element set in $V(G)$ is independent is $(1-p)^{\binom{k}{2}}$. The probability that a fixed k element set in $V(G)$ induces a clique is $p^{\binom{k}{2}}$. Denote the k-element subsets of $V(G)$ by $U_1, U_2, \ldots, U_{\binom{n}{k}}$.

Thus

$$\begin{aligned}
\text{Prob}(\alpha(G) \ge k) &= \text{Prob}(\exists U \subseteq V(G), G[U] = E_k) \\
&\le \text{Prob}((G[U_1] = E_k) \vee (G[U_2] = E_k) \vee \cdots \vee (G[U_{\binom{n}{k}}] = E_k)) \\
&\le \binom{n}{k}(1-p)^{\binom{k}{2}},
\end{aligned}$$

moreover

$$\begin{aligned}
\text{Prob}(\omega(G) \ge k) &= \text{Prob}(\exists U \subseteq V(G), G[U] = K_k) \\
&\le \text{Prob}((G[U_1] = K_k) \vee (G[U_2] = K_k) \vee \cdots \vee (G[U_{\binom{n}{k}}] = K_k)) \\
&\le \binom{n}{k}p^{\binom{k}{2}},
\end{aligned}$$

$\square$

Lemma 110 (11.1.5)**.** Let $G \in \mathcal{G}(n,p)$. Then the expected number of cycles of length k in G is

$$\frac{(n)_k}{2k} p^k,$$

where $(n)_k = n \cdot (n-1) \cdots (n-k+1)$.

Proof. Let $\mathcal{C}_k$ be the set of all cycles of length k in K_n. For a cycle $C \in \mathcal{C}_k$, let $X_C = 1$ if $C \subseteq G$, let $X_C = 0$, otherwise. Let X be the number of cycles of length k in G, i.e., $X = \sum_{C \in \mathcal{C}_k} X_C$. Then $E(X_C) = \mathrm{Prob}(C \subseteq G) = p^k$. Moreover,

$$E(X) = \sum_{C \in \mathcal{C}_k} E(X_C) = |\mathcal{C}_k|\; p^k = \frac{(n)_k}{2k}\; p^k.$$

□

Theorem 9.2 (Erdős)**.** For any $k \geq 2$ there is a graph G on $\sqrt{2}^k$ vertices such that $\alpha(G) < k$ and $\omega(G) < k$. This implies $R(k,k) \geq 2^{k/2}$.

Proof. Let $n = \sqrt{2}^k$ and $G \in \mathcal{G}(n, 1/2)$. Then

$$\mathrm{Prob}((\alpha(G) \geq k) \vee (\omega(G) \geq k)) \leq \mathrm{Prob}(\alpha(G) \geq k) + \mathrm{Prob}(\omega(G) \geq k) \leq 2^{-\binom{k}{2}+1} < 1.$$

Thus $\mathrm{Prob}((\alpha(G) < k) \wedge (\omega(G) < k)) > 0$. Therefore there is a graph G such that $\alpha(G) < k$ and $\omega(G) < k$. □

We need the following standard tool from probability theory.

Theorem 111 (Markov's inequality)**.** Let X be a non-negative random variable and let $t > 0$. Then

$$\mathrm{Prob}(X \geq t) \leq E(X)/t.$$

Theorem 9.3 (Erdős-Hajnal, 11.2.2)**.** For any integer $k \geq 3$ there is a graph with girth greater than k and chromatic number greater than k.

Proof. Fix ϵ, $0 < \epsilon < 1/k$. Let $p = n^{\epsilon - 1}$ and let $G \in \mathcal{G}(n,p)$, $n \geq 1$. Let Y be the number of cycles of length at most k in G. Then

$$E(Y) = \sum_{i=3}^{k} \frac{(n)_i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i \leq \frac{1}{2} k n^k p^k.$$

Here we used the fact that $(np)^i < (np)^k$ for $i < k$, since $np = n^\epsilon \geq 1$. By Markov's inequality,

$$\mathrm{Prob}\left(Y \geq \frac{n}{2}\right) \leq \frac{E(Y)}{n/2} \leq k n^{k-1} p^k \leq k n^{k\epsilon - 1}.$$

Note that $k\epsilon - 1 < 0$, so

$$\text{Prob}\left(Y \geq \frac{n}{2}\right) \overset{n\to\infty}{\to} 0.$$

Consider $\alpha(G)$. We have that

$$\text{Prob}\left(\alpha(G) \geq \frac{n}{2k}\right) < \binom{n}{n/(2k)}(1-p)^{\binom{n/(2k)}{2}}.$$

Thus

$$\text{Prob}\left(\alpha(G) \geq \frac{n}{2k}\right) \overset{n\to\infty}{\to} 0.$$

Choose n sufficiently large so that $\text{Prob}(Y \geq \frac{n}{2}) < 1/2$ and $\text{Prob}(\alpha(G) \geq \frac{n}{2k}) < 1/2$. Thus there is a graph G with at most $n/2$ cycles of length at most k and with $\alpha(G) < \frac{n}{2k}$. Let G' be a graph obtained from G by deleting a vertex from each cycle of length at most k. Then $|V(G')| \geq n/2$, $\alpha(G') \leq \alpha(G) < \frac{n}{2k}$, and G' has girth larger than k. Moreover,

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} > k.$$

Thus G' is the desired graph. □

Lemma 112 (11.3.4)**.** For all $p \in (0,1)$ and $\epsilon > 0$ almost all graphs G in $\mathcal{G}(n,p)$ fulfil

$$\chi(G) > \frac{\log\bigl(1/(1-p)\bigr)}{2+\epsilon} \cdot \frac{n}{\log n}.$$

Remark. Asymptotic behaviour of $\mathcal{G}(n,p)$ for some properties:

- $p_n = \sqrt{2}/n^2 \Rightarrow G$ almost always has a component with > 2 vertices
- $p_n = 1/n \Rightarrow G$ almost always has a cycle
- $p_n = \log n/n \Rightarrow G$ is almost always connected
- $p_n = (1+\epsilon)\log n/n \Rightarrow G$ almost always has a Hamiltonian cycle
- $p_n = n^{-2/(k-1)}$ is the threshold function for containing K_k

In the following we prove several results concerning threshold functions. Before doing so, we need a few more tools from probability theory.

Theorem 113 (Chebyshev's inequality)**.** Let X be a real random variable. Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$. Then

$$\text{Prob}(|X - \mu| \geq t) \leq \sigma^2/t^2,$$

for any $t > 0$.

Theorem 114 (Chernoff's inequality)**.** Let X_i's be independent random variables, $X_i \in \{0,1\}$, $i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$ and let $\mu = E(X)$. Then

$$\mathrm{Prob}(X \le (1-\delta)\mu) \le e^{-\delta^2\mu/2},$$

for any positive δ.

Lemma 115. The threshold function for containing a cycle is $f(n) = 1/n$.

Proof. First assume that $p = o(1/n)$. Let X denote the number of cycles in $G(n,p)$. Then by Markov's inequality $\mathrm{Prob}(G(n,p)$ contains a cycle $) \le E(X) \le \sum_{i\ge 3}(n)_i p^i/(2i) \le (np)^3 \sum_{i\ge 0}(np)^i \le \frac{(np)^3}{1-np} \to_{n\to\infty} 0$.

Now assume that $p = \omega(1/n)$. It is sufficient for us to show that the number of edges in $G(n,p)$ is greater than or equal to n with probability approaching 1 as n goes to infinity.

Assume that n is large enough and $p > (2+\epsilon)/n$, for positive ϵ. Let δ be chosen such that $(1-\delta) = 2/(2+\epsilon)$. Let Y denote the number of edges in $G(n,p)$. Then $\mu := E(Y) = p\binom{n}{2}) \ge (2+\epsilon)/n(n(n-1))/2 = (2+\epsilon)(n-1)/2$. So by Chernoff's inequality $\mathrm{Prob}(Y \le (1-\delta)\mu) \le (1-\delta)\mu) \le e^{-\delta^2\mu/2} \to_{n\to\infty} 0$. Since $(1-\delta)\mu = n-1$, we have that $\mathrm{Prob}(Y \ge n) \to_{n\to\infty} 1$. $\square$

Finally, we consider the threshold for containing a fixed graph H.

Lemma 116. Let H be a fixed graph on v_H vertices and e_H edges. Then n^{-v_H/e_H} is the threshold function for containing H as a subgraph.

Proof. We shall prove one part in general and the second only for $H = K_3$.

Assume first that $p = o(n^{-v_H/e_H})$. Let X denote the number of copies of H in $G(n,p)$. Then $\mu = E(X) = \binom{n}{v_H} c_H p^{e_H} \le c_H n^{v_H} p^{e_H}$, where c_H is a function of H (more specifically, of $Aut(H)$). Then μ approaches 0 as n approaches infinity. This implies that the $\mathrm{Prob}(H \subseteq G(n,p)) \to_{n\to\infty} 0$, by Markov's inequality. This proves the first part of the lemma.

Now, assume that $p = \omega(1/n)$ and $H = K_3$. Assume that $p = d/n$ for sufficiently large d. Let X be the number of K_3's in $G = G(n,p)$ on vertex set $[n]$. Let $X_{i,j,k}$ be a random variable that is 1 if $\{i,j,k\}$ induces K_3 in G, and 0, otherwise. Then $X = \sum_{\{i,j,k\}\in\binom{[n]}{3}} X_{i,j,k}$. We shall estimate the expected value, μ and the variance, σ^2, of X. We have that $\mu = \binom{n}{3}p^3 \approx d^3/6$. Further,

$$E(X^2) = E((\sum_{ijk} X_{i,j,k})^2) = E(\sum_{ijk,i'j'k'} X_{i,j,k}X_{i'j'k'}),$$

where the sums are over all triples and all pairs of triples, respectively, in $\binom{[n]}{3}$ (and where we have abbreviated writing triples $\{i,j,k\}$ as simply ijk).

Let $S_1 = \{\{\{i,j,k\},\{i',j',k'\}\} \in \binom{[n]}{3}^2 : |\{i,j,k\} \cap \{i',j',k'\}| \le 1\}$.

Let $S_2 = \{\{\{i,j,k\},\{i',j',k'\}\} \in \binom{[n]}{3}^2 : |\{i,j,k\} \cap \{i',j',k'\}| = 2\}$.

Let $S_3 = \{\{\{i,j,k\},\{i',j',k'\}\}\} \in \binom{[n]}{3}^2 : |\{i,j,k\} \cap \{i',j',k'\}| = 3\}$.

Then

$$E(X^2) = \sum_{S_1} E(X_{i,j,k}X_{i',j',k'}) + \sum_{S_2} E(X_{i,j,k}X_{i',j',k'}) + \sum_{S_3} E(X_{i,j,k}X_{i',j',k'}),$$

where the sums are over all pairs of triples in S_1, S_2, S_3, respectively. Therefore, $E(X^2) = p^6|S_1| + p^5|S_2| + p^3|S_3| \le p^6\binom{n}{3}^2 + cp^5\binom{n}{5} + p^3\binom{n}{3} \le \mu^2 + o(1) + d^3/6$. Thus $\mathrm{Var}(X) = E(X^2) - E^2(X) \le d^3/6 + o(1)$. So, using Chebyshev's inequality we obtain

$$\mathrm{Prob}(X=0) \le \mathrm{Prob}(|X - E(X)| \ge E(X)) \le \mathrm{Var}(X)/E(X)^2 \le (d^3/6 + o(1))/(d^6/36).$$

The last term above is at most $6/d^3 + o(1)$. If $d = d(n)$ approaches infinity as n goes to infinity, we have that $\mathrm{Prob}(X \ne 0) \to_{n\to\infty} 1$. $\square$

Lemma 9.4 (Lovász Local Lemma)**.** Let $A_1, \ldots, A_n$ be events in some probabilistic space. If $\mathrm{Prob}(A_i) \le p \in (0,1)$, each A_i is mutually independent from all but at most $d \in \mathbb{N}$ A_is and $ep(d+1) \le 1$, then

$$\mathrm{Prob}\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

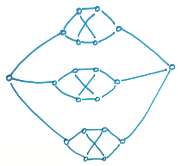
Lemma 117. The *Van-der-Waerden's number* $W(k)$ is the smallest n such that any 2-coloring of $[n]$ contains a monochromatic arithmetic progression of length k. We can prove $W(k) \ge 2^{k-1}/(ek^2)$ with the Lovász Local Lemma.

Theorem 118 (Erdős-Rényi, 1960)**.** Let H be a graph with at least one edge. Let $\epsilon'(H) = \max\{|E(H')|/|V(H')| : H' \subseteq H\}$. Then $t(n) = n^{-1/\epsilon'(H)}$ is a threshold function for a property $\mathcal{P} = \{G : H \subseteq G\}$.

Theorem 119 (Bollobás-Thomason, 1987)**.** There is a threshold function for any increasing graph property, i.e., a property that is closed under taking supergraphs.

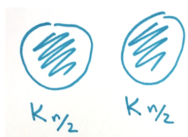
10 Hamiltonian cycles

Lemma 10.1 (Necessary condition for the existence of a Hamiltonian cycle)**.** If G has a Hamiltonian cycle, then for every non-empty $S \subseteq V$ the graph $G - S$ cannot have more than $|S|$ components.

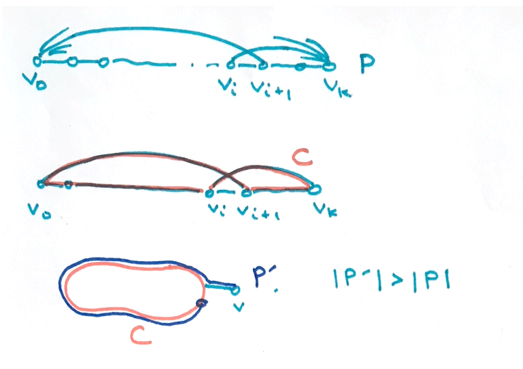


Non-hamiltonian graph.

Theorem 10.2 (Dirac, 10.1.1)**.** Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.



Proof. First we note that G is connected, otherwise a smaller component has all vertices of degree at most $n/2 - 1$. Consider a longest path $P = (v_0, \ldots, v_k)$. Then $N(v_0), N(v_k) \subseteq V(P)$. Since $|N(v_0)|, |N(v_k)| \geq n/2$, and $k \leq n - 1$, we have by pigeonhole principle that $v_0 v_k \in E(G)$ or there is i, $0 < i < k-1$ such that $v_0 v_{i+1} \in E(G)$ and $v_i v_k \in E(G)$. In any case there is cycle C on $k+1$ vertices in G. If $k + 1 = n$, C is a Hamiltonian cycle and we are done. If $k + 1 < n$, since G is connected there is a vertex v not in C that is adjacent to a vertex with C. Then v and C induce a graph that contains a spanning path, i.e. a path on $k + 2$ vertices, a contradiction to maximality of P. □

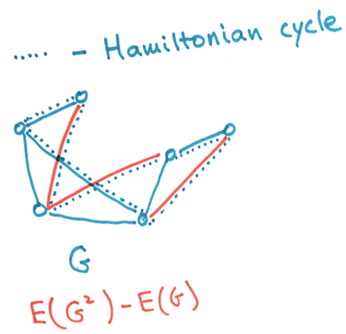


Theorem 120. Every graph on $n \geq 3$ vertices with $\alpha(G) \leq \kappa(G)$ is Hamiltonian.

Theorem 121 (Tutte, 10.1.4)**.** Every 4-connected planar graph is Hamiltonian.

Definition. Let $G = (V, E)$ be a graph. The *square* of G, denoted by G^2, is the graph $G^2 := (V, E')$ with $E' := \{uv : u, v \in V, d_G(u, v) \leq 2\}$. square, G^2

Theorem 122 (Fleischner's Theorem, 10.3.1)**.** If G is 2-connected, then G^2 is Hamiltonian.



We say that an integer sequence $(b_1, b_2, \ldots, b_n)$ is *pointwise greater* than an integer sequence $(a_1, a_2, \ldots, a_n)$ if $a_i \leq b_i$ holds for all $1 \leq i \leq n$. We call an integer sequence $(a_1, a_2, \ldots, a_n)$ a *Hamiltonian sequence* if every graph on n vertices with degree sequence pointwise greater than $(a_1, a_2, \ldots, a_n)$ is Hamiltonian.

Theorem 10.3 (Chvátal, 10.2.1)**.** An integer sequence $(a_1, a_2, \ldots, a_n)$ with $0 \leq a_1 \leq \cdots \leq a_n < n$ and $n \geq 3$ is Hamiltonian if and only if $a_i \leq i$ implies $a_{n-i} \geq n - i$ for all $i < n/2$.

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